Economics 202A
Lecture Outline #7 (version 2.1)

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Optimal Consumption in a Frictionless World: Complete Markets

To understand consumption under uncertainty, we start with the benchmark case of *complete markets.* Complete asset markets effectively allow consumers to buy insurance against any contingency (or to sell insurance). This is possible because there exist assets with returns differentiated across every state of nature, and, subject to an overall budget constraint, individuals can purchase any (positive or negative) amount of such assets.

This is not realistic — although one way to read the proliferation of exotic derivative products in recent years is as an evolution of real-world markets toward the ideal of completeness.

Why, then, consider this case? Because the availability of this benchmark — like the hypothetical "frictionless plane" in physics — allows us to get a handle on more complex problems. For example, Newton's law $F = ma$ is counterintuitive until one learns to abstract from the force exerted by friction.

Assumptions. Let's start with a pure endowment model (no investment or production). There are two periods. On date 1, individual i's endowment is y^i. From the perspective of date 1, however, the date 2 endowment is a random variable. There are also only two possible states of nature on date 2. In state 1 the endowment is $y^i(1)$, in state 2 it is $y^i(2)$.

Let c^i denote the individual's date 1 consumption, $c^i(1)$ and $c^i(2)$ the individual's *contingency plans* for consumption on date 2. The plans are contingent on the state that actually occurs on date 2. The probability that state s occurs is $\pi(s)$, where, summing over all states s, $\Sigma_s \pi(s) = 1$.

A key hypothesis is that the individual chooses the consumption plan that maximizes average lifetime utility,

$$\begin{aligned} U^i &= \pi(1)\left\{u(c^i) + \beta u\left[c^i(1)\right]\right\} + \pi(2)\left\{u(c^i) + \beta u\left[c^i(2)\right]\right\} \\ &= u(c^i) + \beta\left\{\pi(1)u\left[c^i(1)\right] + \pi(2)u\left[c^i(2)\right]\right\} \\ &= u(c^i) + \beta \mathrm{E}u\left[c^i(s)\right], \end{aligned}$$

where $c(s)$ denotes consumption in state s. This is the von Neumann-Morgenstern expected utility criterion and, being linear in probabilities, it is somewhat special. One of its consequences (as we shall see) is that it forces the intertemporal substitution elasticity to equal the (inverse) coefficient of absolute risk aversion for isoelastic utility. We shall define the risk aversion coefficient later.

A basic *Arrow-Debreu security* for state s pays its owner 1 unit of output on date 2 if state s occurs and nothing otherwise. (In contrast, a riskless *bond* pays its owner the same amount of output in every state.)

Let r be the rate of interest on a bond. We define r by the definition that $1/(1+r)$ is the price (all prices are in terms of date 1 consumption) of a bond paying its owner 1 unit of output on date 2 regardless of the state of nature. We further define

$$\frac{p(s)}{1+r} = \text{ date 1 price of the Arrow-Debreu state } s \text{ security.}$$

Suppose you were to buy exactly one Arrow-Debreu security for each possible state s. What would we call this "bundle" of assets, which pays you exactly 1 unit of output on date 2 regardless of the state? The name is bond. Thus we have the arbitrage relation:

$$\sum\nolimits_s p(s) = 1.$$

Think of there as being three goods in the model — date 1 consumption and date 2 consumption contingent on state of nature. The Arrow-Debreu assets' prices define the prices of future contingent consumptions. So individual i maximizes U^i subject to the lifetime budget constraint

$$c^i + \frac{p(1)}{1+r}c^i(1) + \frac{p(2)}{1+r}c^i(2) = y^i + \frac{p(1)}{1+r}y^i(1) + \frac{p(2)}{1+r}y^i(2). \qquad (1)$$

Individual choice. As in our prior, deterministic model people smooth consumption across dates (subject to intertemporal price incentives), but they also plan to have smooth consumption across states — subject to interstate price incentives.

We see how this works by writing down the usual Lagrangian for maximizing U^i subject to (1) and finding the first-order conditions:

$$\begin{aligned} u'(c^i) &= \lambda^i, \\ \beta\pi(s)\,u'\left[c^i(s)\right] &= \lambda^i \frac{p(s)}{1+r}. \end{aligned}$$

Combine these to get

$$u'(c^i)\frac{p(s)}{1+r} = \beta\pi(s)\,u'\left[c^i(s)\right],$$

the Euler equation for the state-s Arrow-Debreu security. Interpretation: At an optimum, the preset utility forgone by buying the asset just equals the future utility it is expected to yield. The conditions (just add them up) also imply the *stochastic* Euler equation for bonds,

$$u'(c^i) = (1+r)\beta \mathrm{E} u'\left[c^i(s)\right].$$

Notice that the ratio of marginal utilities across states on date 2 is

$$\frac{u'\left[c^i(1)\right]}{u'\left[c^i(2)\right]} = \frac{p(1)/\pi(1)}{p(2)/\pi(2)}.$$

When $p(s) = \pi(s)$, we say that prices are *actuarially fair.* In general they need not be, in which case people will not elect to insure their consumption completely (arrange for equal consumption in every state of nature). In general, the prices $p(s)$ will reflect not only the state probabilities $\pi(s)$, but also the aggregate output levels in various states, with $p(s)/\pi(s)$ being relatively higher in states where aggregate output is relatively scarce.

There are some important implications for the comovements of individual consumption levels over time. If individuals face common prices and have common probability assessments $\pi(s)$ and discount factors β, then for any consumers i and j, and for any state s,

$$\frac{u'\left[c^i(s)\right]}{u'(c^i)} = \frac{u'\left[c^j(s)\right]}{u'(c^j)}.$$

For the isoelastic utility function, this implies

$$\log\left[c^i(s)/c^i\right] = \frac{\sigma_i}{\sigma_j}\log\left[c^j(s)/c^j\right].$$

Thus consumption growth rates are perfectly correlated. Studies of micro-data tend to reject this implication of complete markets.

Applications of Arrow-Debreu prices. AD prices are useful in a complete-markets setting because they give a market valuation of output available in various states. We then can value contingent output as we would any other good. Applications include investment under uncertainty and asset pricing.

Regarding investment, imagine that future output is given by $A(s)F(K)$, where K is capital accumulated prior to production [and the realization of the productivity shock $A(s)$]. One unit of output translates into one unit of installed capital (contrary to the q model to be discussed later) and capital depreciates at rate δ. Under certainty the rule for optimal capital would be $1 = (1+r)^{-1}[AF'(K) + 1 - \delta]$. (Why?) Under uncertainty with complete markets, we can simply add up the capital's possible future products state by state and price those using the AD prices:

$$1 = \sum\nolimits_s \frac{p(s)}{1+r}[A(s)F'(K) + 1 - \delta].$$

Next suppose we have an asset that pays a dividend $d(s)$ in state s. If we are in a two-period world (so that asset value is zero after the dividend pay-out), the asset price is given simply by

$$q = \sum\nolimits_s \frac{p(s)}{1+r} d(s).$$

Using the Euler equation for AD securities, we can alternatively write this as

$$q = \beta \sum\nolimits_s \frac{\pi(s)u'\left[c(s)\right]}{u'(c)} d(s),$$

which can be re-written as the asset Euler equation

$$u'(c)q = \beta \mathrm{E}\left\{u'[c(s)]d(s)\right\} \Leftrightarrow q = \mathrm{E}\left\{\frac{\beta u'[c(s)]}{u'(c)} d(s)\right\}$$

(Whose intertemporal marginal rate of substitution $\beta u'[c(s)]/u'(c)$ are we using above? Does it matter?)

For a long-lived asset in a economy with more time periods we would instead have

$$q_t = \mathrm{E}_t\left\{\frac{\beta u'(c_{t+1})}{u'(c_t)}\left(d_{t+1} + q_{t+1}\right)\right\},$$

a stochastic difference equation in q_t. The intertemporal marginal rate of substitution $\beta u'(c_{t+1})/u'(c_t)$ is also called the *pricing kernel*.

Optimal Consumption with Incomplete Markets

Let us analyze, more generally, a situation where asset markets mat be incomplete.

To lead in to the permanent income/life-cycle discussion, I now assume an infinite horizon.

Dynamic programming of consumption and portfolio choice. The consumer maximizes expected lifetime utility beginning at date $t = 0$,

$$\mathrm{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t u(c_t) \right\}.$$

There are N risky assets with random net real returns r_t^i between the end of date t and start of $t+1$. The individual enters t with financial assets a_t, receives wages w_t, and consumes c_t. Assets plus new savings $a_t + w_t - c_t$ are then allocated among the N available assets, with x_t^i denoting the portfolio share of the i^{th} asset. The gross payoffs on the portfolio sum to assets at the start of $t+1$, a_{t+1}. The implied constraints are a given initial asset level a_0 plus:

$$a_{t+1} = \sum_{i=1}^{N} x_t^i (1 + r_{t+1}^i)(a_t + w_t - c_t),$$

$$\sum_{i=1}^{N} x_t^i = 1.$$

Let $V(a_t)$ denote the value function at the start of period t.[1] The Bellman equation for the problem is the recursive relationship

$$V(a_t) = \max_{c_t, x_t^i} \left\{ u(c_t) + \beta \mathrm{E}_t V(a_{t+1}) \right\},$$

[1] More generally, if wages follow a Markov process, current and possibly past wages would appear as additional state variables in the value function. Because wages are not chosen by the consumer, however, I simplify the notation by suppressing the dependence of the value function on the wage process.

where the maximization is done subject to the preceding two constraints.

To derive the first-order conditions for a maximum, set up the Lagrangian

$$u(c_t) + \beta \mathrm{E}_t \left\{ V \left[\sum_{i=1}^{N} x_t^i (1 + r_{t+1}^i)(a_t + w_t - c_t) \right] \right\} - \lambda \left(\sum_{i=1}^{N} x_t^i - 1 \right).$$

The first-order conditions for a maximum are

$$u'(c_t) - \beta \mathrm{E}_t \left\{ \sum_{i=1}^{N} x_t^i (1 + r_{t+1}^i) V'(a_{t+1}) \right\} = 0$$

and, for all assets i,

$$\beta \mathrm{E}_t \left\{ (1 + r_{t+1}^i) V'(a_{t+1}) \right\} (a_t + w_t - c_t) - \lambda = 0.$$

Multiply the last condition by x_t^i (which is known as of date t, because it is chosen then) and sum over $i = 1, ..., N$. The implication is that $u'(c_t) = \lambda / \left(a_t + w_t - c_t \right)$. As a result, by the envelope condition

$$u'(c_{t+1}) = V'(a_{t+1}),$$

we find that for every available asset i,

$$u'(c_t) = \beta \mathrm{E}_t \left[(1 + r_{t+1}^i) u'(c_{t+1}) \right].$$

So an Euler equation holds for each asset even if markets are incomplete and human capital is not tradable.

Quadratic case: Hall's random walk hypothesis. Let there be an asset with the riskless real return r. Its Euler equation is

$$u'(c_t) = (1 + r) \beta \mathrm{E}_t \left[u'(c_{t+1}) \right].$$

Assume that $u(c_t)$ has the quadratic form

$$u(c_t) = a c_t - \frac{b}{2} c_t^2$$

and that $(1 + r)\beta = 1$. (Quadratic utility is at best an approximation; taken literally and globally, it would imply the possibility of negative marginal utility of consumption.) Because $u'(c) = a - bc$, the Euler equation implies Hall's random-walk hypothesis:

$$c_t = \mathrm{E}_t c_{t+1}.$$

Hall's basic idea is to test this relationship rather than to estimate a structural consumption function.

A key implication is that consumption should respond to unexpected news, but not to predictable events. The reason is that technically speaking, consumption is a martingale (a special case of a random walk). Thus, we can write the consumption process as

$$c_{t+1} = c_t + u_{t+1}$$

where u_{t+1} is uncorrelated with any information known as of date t.

This is the key implication being tested in Hsieh's paper on the Alaska fund in the *AER* (see the 202A reading list). He finds that Alaska fund oil dividends, which are substantial and quite predictable as to amounts and timing, do not affect Alaskans' consumption when they are paid out.

The certainty-equivalent consumption function. Consider a world in which risk-free bonds are the only asset. Ex post, and with an infinite horizon, consumption must satisfy the intertemporal constraint

$$\sum_{t=0}^{\infty} \frac{c_t}{(1+r)^t} = a_0 + \sum_{t=0}^{\infty} \frac{w_t}{(1+r)^t}.$$

Ex ante, we there fore have

$$\mathrm{E}_0 \sum_{t=0}^{\infty} \frac{c_t}{(1+r)^t} = a_0 + \mathrm{E}_0 \sum_{t=0}^{\infty} \frac{w_t}{(1+r)^t}.$$

Because $\mathrm{E}_0 c_t = \mathrm{E}_0 \mathrm{E}_{t-1} c_t = \mathrm{E}_0 c_{t-1} = \mathrm{E}_0 c_{t-2} = ... = c_0$, we can solve for c_0:

$$c_0 = \frac{r a_0}{1+r} + \frac{r}{1+r} \mathrm{E}_0 \sum_{t=0}^{\infty} \frac{w_t}{(1+r)^t}.$$

This formulation gets at Milton Friedman's idea of "permanent income" as a determinant of consumption: the present value of wage income is what matters in the consumption function (along with the interest yield on financial wealth). Accordingly, permanent changes in wages will have a bigger effect on consumption than will transitory changes. The life-cycle hypothesis of Franco Modigliani and Richard Brumberg is motivated by similar economics, but accounts for the typical lifetime income cycle. The age-earnings profile is usually positively sloped, then flattens out, then drops sharply with retirement. Accordingly, workers will tend to dissave while young, pay back debt and accumulate wealth during prime earning years, then retire on savings and accumulated pension benefits.

Precautionary saving behavior. The certainty equivalent model contains no true role for risk. As an alternative consider the utility function

$$u(c) = \frac{c^{1-R} - 1}{1 - R}.$$

The expression

$$-\frac{cu''(c)}{u'(c)} = R$$

is known as the Arrow-Pratt coefficient of relative risk aversion. Of course, it is also $1/\sigma$, where σ is the intertemporal substitution elasticity — an equivalence that is sometimes unfortunate but that can be relaxed with more general utility specifications.

With $\beta = (1+r)^{-1}$, assume also that the distribution of $\log c_{t+1}$ is normal from the perspective of date t. That is, assume that

$$\log c_{t+1} \sim N(\mathrm{E}_t \log c_{t+1}, \sigma_t^2).$$

By the properties of the lognormal distribution, the Euler equation is

$$\begin{aligned} c_t^{-R} &= \mathrm{E}_t c_{t+1}^{-R} \\ \Leftrightarrow \quad c_t^{-R} &= e^{-R\mathrm{E}_t \log c_{t+1} + \frac{R^2}{2}\sigma_t^2}. \end{aligned}$$

Because $c_t = e^{\log c_t}$, taking logs of both sides gives

$$\log c_t = \mathrm{E}_t \log c_{t+1} - \frac{R}{2}\sigma_t^2.$$

So here we have an effect of consumption *variance* on the level of consumption: higher variance lowers consumption today, and therefore increases saving. This precautionary saving effect is proportional to the measure of risk aversion, R.

More generally, the Euler equation in this case reads

$$u'(c_t) = \mathrm{E}_t u'(c_{t+1}).$$

A mean-preserving expansion in the variance of c_{t+1} must raise $\mathrm{E}_t u'(c_{t+1})$ if $u'(c)$ is a strictly convex function of c, that is, if the third derivative $u'''(c) > 0$. (This follows from Jensen's inequality.) If $\mathrm{E}_t u'(c_{t+1})$ rises, so does $u'(c_t)$, which means that c_t falls and saving rises. So a positive third derivative of utility leads to precautionary saving. For the quadratic utility function, $u'''(c) = 0$, so there is no precautionary saving in that case.

Another way to see the impact of higher consumption variability on $\mathrm{E}u'(c)$ is through a second-order approximation. Let $\bar{c} \equiv \mathrm{E}c$. Then, taking a Taylor approximation around $c = \bar{c}$ gives us

$$u'(c) \approx u'(\bar{c}) + u''(\bar{c})(c - \bar{c}) + \tfrac{1}{2}u'''(\bar{c})(c - \bar{c})^2.$$

Taking expected values lead to

$$\mathrm{E}u'(c) \approx u'(\bar{c}) + \tfrac{1}{2}u'''(\bar{c})\mathrm{Var}(c).$$

Thus, when (and only when) the third derivative u''' is positive, a rise in the variance of consumption $\mathrm{Var}(c)$, holding the expected level $\bar{c}$ constant, raises $\mathrm{E}u'(c)$.

Economics 202A, Problem Set 4

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1. *Interest rates and consumption.* An individual has the exponential period utility function

$$u(C) = -\gamma \mathrm{e}^{-C/\gamma}$$

($\gamma > 0$) and maximizes

$$u(C_t) + \beta u(C_{t+1})$$

($0 < \beta < 1$) subject to the budget constraint

$$C_t + RC_{t+1} = Y_t + RY_{t+1} \equiv W_t$$

[where $R = (1+r)^{-1}$, so that a fall in the real interest rate r means a rise in the market discount factor R].

(a) Solve for C_{t+1} as a function of C_t, R, and β using the consumer's intertemporal Euler equation.

(b) What is the optimal level of C_t, given W_t, R, and β? [In other words, solve for the date t consumption function.]

(c) By differentiating your consumption function (including W_t) with respect to R, show that:

$$\frac{\mathrm{d}C_t}{\mathrm{d}R} = -\frac{C_t}{1+R} + \frac{Y_{t+1}}{1+R} + \frac{\gamma}{1+R}\left[1 - \log(\beta/R)\right].$$

[Hint: Your consumption function has the form $C = f(W,R)/(1+R)$.Therefore,

$$\frac{\mathrm{d}C}{\mathrm{d}R} = -\frac{C}{1+R} + \frac{1}{1+R}\left[\frac{\partial f}{\partial W}\frac{\mathrm{d}W}{\mathrm{d}R} + \frac{\partial f}{\partial R}\right],$$

which gives you half the answer.]

(d) What is the intertemporal substitution elasticity for the exponential utility function? [Calculate this elasticity at an allocation where $C_t = C_{t+1} = \overline{C}$. It is a function $\sigma(\overline{C})$ of $\overline{C}$, not a constant.]

(e) Show that the derivative calculated in part (c) above can be expressed as

$$\frac{\mathrm{d}C_t}{\mathrm{d}R} = \frac{\sigma(C_{t+1})C_{t+1}}{1+R} + \frac{Y_{t+1} - C_{t+1}}{1+R}.$$

(f) Explain intuitively why, for someone with $Y_{t+1} > C_{t+1}$, a rise in R (that is, a fall in the real interest rate r), unambiguously raises consumption on date t.

2. *Optimal consumption with incomplete markets.* A consumer has the quadratic period utility function $u(C) = \alpha C - (\gamma/2)\, C^2$ and maximizes $u(C_t) + \beta u(C_{t+1})$ subject to the constraints

$$A_{t+1} = (1+r)A_t + Y_t - C_t, \quad A_t \text{ given,}$$

$$C_{t+1} = (1+r)A_{t+1} + Y_{t+1}(\mathfrak{s}), \quad \mathfrak{s} \in \{1, 2, ..., S\}\,.$$

Here, r is the real rate of interest (so that $1/(1+r)$ is the current price of a unit of output delivered next period with probability 1). Let $\pi(\mathfrak{s})$ be the probability of state of nature $\mathfrak{s}$ from the perspective of date t and assume that $\beta = 1/\,(1+r)$.

(a) Ignore for the moment the constraint that date $t+1$ consumption be nonnegative. Compute and interpret the optimal level of C_t.

(b) Suppose the consumer has an infinite horizon and there is uncertainty over output on all future dates. Use your answer to (a) to guess the consumption function and use the "random walk" result to prove that your guess is correct.

(c) Let's return to the 2-period case in part (a) but now take seriously the constraint that

$$C_{t+1}(\mathfrak{s}) \geq 0, \quad \forall \mathfrak{s}.$$

Renumber the states of nature $\mathfrak{s}$ (if necessary) so that

$$Y_{t+1}(1) = \min_{\mathfrak{s}} \{Y_{t+1}(\mathfrak{s})\}\,.$$

Show the following: If

$$(1+r)A_t + Y_t - \mathrm{E}_t\{Y_{t+1}\} > -(2+r)Y_{t+1}(1)/(1+r),$$

then the result in (a) still holds. (Why, intuitively?) Otherwise, the consumption function is:

$$C_t = (1 + r)A_t + Y_t + Y_{t+1}(1)/(1 + r).$$

(d) Still thinking about the 2=period case, suppose the consumer faces *complete* asset markets such that $p(\mathfrak{s})$, the price in terms of sure date $t + 1$ output of a state-$\mathfrak{s}$ contingent unit of date $t + 1$ output, equals $\pi(\mathfrak{s})$. Compute the optimal value of C_t. Do we gave to worry now about the non-negativity constraint on date $t + 1$ consumption?