Economics 202A
Lecture Outline #5 (version 1.3)

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Endogenous Growth

We have already seen one crude endogenous growth model, the so-called "AK" model. It is crude because it does not give a realistic account of the channels through which productivity grows over time – namely, innovation and the creation of new knowledge.

We now turn to a class of models that indeed endogenize the innovative process. The challenge in thinking about these problems is that the creation of knowledge, which has a public-good aspect, is different from the production of other economic goods.

The endogenous growth literature began with contributions of Robert Lucas and especially Paul Romer in the 1980s and 1990s, although the ideas certainly had important precursors in the growth literature of the 1960s.

A Model of Endogenous Growth: The Basic Idea

The model builds on some of the ideas about differentiated products that also underlie the "new trade theory" developed by Paul Krugman and others in the late 1970s and early 1980s. In the model, additional "varieties" of differentiated capital goods will boost productivity, and the process through which new capital goods are invented is endogenized.

In this economy, production of a *final* consumption good is given by

$$Y_t = F(K_{1,t}, ..., K_{A_t,t}, L_{Y,t}) = \left(\sum_{j=1}^{A_t} K_{j,t}^{\alpha} \right) L_{Y,t}^{1-\alpha} = \sum_{j=1}^{A_t} K_{j,t}^{\alpha} L_{Y,t}^{1-\alpha},$$

where $L_{Y,t}$ is the amount of labor employed in the final goods sector at t and $j \in \{1, 2, ..., A_t\}$ indexes the different types of capital that can be used in production as of t. Labor not devoted to final-goods production will, as we shall see, be devoted to research and development into new capital goods.

We assume that the capital depreciation rate is $\delta = 1$, so that the price of a machine is its rental rate.

Note some interesting features of this production setup. At any point in time, there are constant returns to scale with respect to the existing factors of production, no matter how many there are. But while the marginal product of an existing capital good is finite, the marginal product of a *new* capital good is infinite.

A different thought experiment gives a good illustration of why the preceding production function can generate endogenous growth. Imagine combining 1 unit each of N capital goods with 1 unit of Labor; we get $Y = N$. Instead, imagine we combine $N/(N+1)$ units each of $N+1$ capital goods with 1 unit of labor. We get

$$\begin{aligned} Y &= \sum_{j=1}^{N+1} \left(\frac{N}{N+1}\right)^{\alpha} = (N+1)\left(\frac{N}{N+1}\right)^{\alpha} \\ &= N^{a}(N+1)^{1-\alpha} > N. \end{aligned}$$

So with more capital goods, the output/labor ratio rises holding constant the *amount* of capital input (measured in terms of consumption goods). Thus, the creation of new capital goods has the potential to raise productivity and per-worker output over time.

Notice, finally, that if $K_{j,t} = \tilde{K}_t$ for all varieties j (as is the case in equilibrium when all goods are symmetric), then

$$Y_t = \sum_{j=1}^{A_t} \tilde{K}_t^{\alpha} L_{Y,t}^{1-\alpha} = A_t \tilde{K}_t^{\alpha} L_{Y,t}^{1-\alpha} = \tilde{K}_t^{\alpha}\left(A_t^{\frac{1}{1-\alpha}} L_{Y,t}\right)^{1-\alpha},$$

so the production side looks equivalent to what we assumed for the Solow model. What we will add, as we now show, is a model of how A_t grows endogenously over time.

Production of Capital Goods and Blueprints for New Goods

To produce one unit of capital (of any kind) you need exactly one unit of final output. Capital goods are produced by monopolistic firms. To set up a firm you need to purchase a blueprint for the specific variety j of capital good you will produce. (The cost of the blueprint is sunk.) You can then use a unit of output on date t to yield a unit of your capital good j on date $t+1$, which you sell (rent) at price p_j.

We will assume that more labor devoted to research and development (R&D) results in an expanded set of blueprints allowing the production of more varieties of capital. Specifically, if L_A is labor input to the R&D sector,

$$A_{t+1} - A_t = \theta A_t L_{A,t}. \tag{1}$$

According to eq. (1), labor productivity in R&D is proportional to the existing stock of "knowledge" – so in effect, we have learning by doing. This assumption captures the important idea that, as a public good, new knowledge is *nonrival* (more than one person can use it at the same time) and *nonexcludable* (people cannot be prevented from using knowledge). The learning by doing is external to firms; each firm in R&D behaves competitively.[1]

A blueprint can be put into use the very same period in which it is developed. The total labor force L is constant and fully employed,

$$L = L_Y + L_A.$$

Solving the Model: First Steps

The key is to figure out how the labor force is divided between final-goods production and R&D. The more labor goes into R&D, the faster the growth rate of the economy. The level of output of blueprints, in turn, depends on their price in terms of final goods, p_A.

Let us *conjecture* that in equilibrium we will observe a constant real rate of interest r, constant relative prices, a constant demand for each type of capital, and a constant allocation of labor to sectors of the economy. (Later we show that these guesses are all correct.) Let us start by considering the demand of final-goods firms for capital goods, given by the solution to

$$\max_{\{K_j\}} \sum_{j=1}^{A_t} K_j^{\alpha} L_Y^{1-\alpha} - \sum_{j=1}^{A_t} p_j K_j - w L_Y,$$

where p_j (once again) is the output price of capital of type j and w is the wage in terms of final output. The first-order condition for a maximum for K_j is

$$p_j = \alpha K_j^{\alpha-1} L_Y^{1-\alpha}. \tag{2}$$

[1] Think of the R&D sector as consisting of individual small competitive firms, each with the production function

$$\text{flow of new blueprints} = \theta A_t \ell_{A,t},$$

where $\ell_{A,t}$ is the firm's labor input on date t and A_t is the economy's stock of exisiting blueprints. The individual firm takes $\{A_t\}$ to be exogenous to its decisions.

Thus, the demand for a capital good is[2]

$$K_j = \left(\frac{\alpha}{p_j}\right)^{\frac{1}{1-\alpha}} L_Y.$$

What does this imply for producers of the intermediate capital goods? The (intertemporal) profits of intermediate producer j are

$$\Pi_j = \frac{p_j K_j}{1+r} - K_j = \frac{\alpha K_j^{\alpha} L_Y^{1-\alpha}}{1+r} - K_j.$$

Maximizing Π_j with respect to K_j yields:

$$\frac{\alpha^2 K_j^{\alpha-1} L_Y^{1-\alpha}}{1+r} - 1 = 0$$

or

$$\bar{K} = \left(\frac{\alpha^2}{1+r}\right)^{\frac{1}{1-\alpha}} \bar{L}_Y$$

(where the j subscript has been dropped, as all capital goods are symmetric). Substituting this equation into eq. (2) yields the (constant) relative price of a (generic) intermediate capital good:

$$\begin{aligned}
\bar{p} &= \alpha \bar{K}^{\alpha-1} \bar{L}_Y^{1-\alpha} \\
&= \alpha \left[\left(\frac{\alpha^2}{1+r}\right)^{\frac{1}{1-\alpha}} \bar{L}_Y\right]^{\alpha-1} \bar{L}_Y^{1-\alpha} \\
&= \frac{1+r}{\alpha}.
\end{aligned}$$

For a constant elasticity demand function, a standard result is that a monopolist's price is a constant markup over cost.[3] Here we see that

$$\frac{\text{Price}}{\text{Cost}} = \frac{\bar{p}}{1+r} = \frac{1}{\alpha} = \frac{\frac{1}{1-\alpha}}{\frac{1}{1-\alpha} - 1}.$$

The cost of production is 1 on date $t-1$, and the price obtained (also from the perspective of date $t-1$) is $\bar{p}/(1+r)$.

[2] We also see that $(1-\alpha)\sum_{j=1}^{A_t} K_j^{\alpha} L_Y^{-\alpha} = w$.

[3] If the price elasticity of demand is η, the markup is $\eta/(\eta-1)$, which goes to 1 as $\eta \to \infty$. In the present model, $\eta = 1/(1-\alpha)$.

Given all this, what is the profit that a capital-good producer earns? We need to know this because the requirement that the stream of profits covers sunk cost is what ties the model down. Substitution yields:

$$\begin{aligned}\bar{\Pi} &= \frac{\bar{p}\bar{K}}{1+r} - \bar{K} = \left(\frac{\bar{p}}{1+r} - 1\right)\left(\frac{\alpha^2}{1+r}\right)^{\frac{1}{1-\alpha}} \bar{L}_Y \\ &= \left(\frac{1-\alpha}{\alpha}\right)\left(\frac{\alpha^2}{1+r}\right)^{\frac{1}{1-\alpha}} \bar{L}_Y. \end{aligned} \tag{3}$$

There is free entry into producing intermediate goods, so the price of a blueprint must equal the present discounted value of $\bar{\Pi}$ above, or

$$\begin{aligned}\bar{p}_A &= \sum_{t=0}^{\infty} \frac{\bar{\Pi}}{(1+r)^t} = \frac{1+r}{r}\bar{\Pi} \\ &= \frac{1+r}{r}\left(\frac{1-\alpha}{\alpha}\right)\left(\frac{\alpha^2}{1+r}\right)^{\frac{1}{1-\alpha}} \bar{L}_Y \end{aligned} \tag{4}$$

An important point: if we did not have monopoly in the capital-producing sector, there would be no stream of monopoly profits to cover the sunk cost of blueprints, and so blueprints would never be purchased. In the market setting we have assumed, monopoly – and some degree of monopoly inefficiency – is necessary to sustain positive growth.

Equilibrium Growth Rate

Equilibrium growth in the number of capital goods is given by

$$g = \frac{A_{t+1} - A_t}{A_t} = \theta \bar{L}_A.$$

Production of each specific capital good will remain constant at $\bar{K}$.

What ties down the equilibrium allocation of labor, and hence g, is the preceding eq. (4) for $\bar{p}_A$. Suppose there are too many workers in final goods production (relative to the equilibrium) because workers are paid more in final goods than in R&D. Then the demand for capital (to equip those workers) will be high, raising the profits of intermediate producers and causing them to bid up the price of blueprints $\bar{p}_A$. But that development, in turn will raise the wages paid in the R&D sector, drawing workers out of final goods. The process will continue until wages in the two sectors are equal.

We formalize the requirement that workers have the same marginal value product in both sectors by requiring that

$$\text{MVPL in R\&D} = \bar{p}_A \theta A = (1-\alpha) L_Y^{-\alpha} \sum_{j=1}^{A} \bar{K}^{\alpha} = (1-\alpha) A \bar{K}^{\alpha} L_Y^{-\alpha} = w.$$

The solution is

$$\begin{aligned}
\bar{L}_Y &= \left[\frac{(1-\alpha)}{\bar{p}_A \theta}\right]^{\frac{1}{\alpha}} \bar{K} \\
&= \left[\frac{r(1-\alpha)}{(1+r)\bar{\Pi}\theta}\right]^{\frac{1}{\alpha}} \bar{K} \\
&= \left[\frac{r(1-\alpha)}{(1+r)\left(\frac{1-\alpha}{\alpha}\right)\left(\frac{\alpha^2}{1+r}\right)^{\frac{1}{1-\alpha}} \bar{L}_Y \theta}\right]^{\frac{1}{\alpha}} \left(\frac{\alpha^2}{1+r}\right)^{\frac{1}{1-\alpha}} \bar{L}_Y \Rightarrow \\
1 &= \left[\frac{\alpha r}{(1+r)\bar{L}_Y \theta}\right]^{\frac{1-\alpha}{\alpha}} \left(\frac{\alpha^2}{1+r}\right)^{-\frac{1-\alpha}{\alpha}} \Rightarrow \\
1 &= \left[\frac{\alpha r}{(1+r)\bar{L}_Y \theta}\right] \left(\frac{\alpha^2}{1+r}\right)^{-1} \Rightarrow \\
\bar{L}_Y &= \frac{r}{\alpha\theta}.
\end{aligned}$$

This is consistent, by the way, with the assumption we made that $\bar{L}_Y$ is constant. We can now also find the long-run rate of growth, which is

$$\bar{g} = \theta \bar{L}_A = \theta(L - \bar{L}_Y) = \theta L - \frac{r}{\alpha}. \tag{5}$$

Notice that there is a "scale effect" here: a bigger work force implies more innovation and hence faster growth. Higher interest rates retard growth – though we have yet to solve for the equilibrium rate of interest r.

Let's do so next. If the lifetime utility function of the representative consumer is

$$U_0 = \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}},$$

then the intertemporal Euler equation is

$$C_t^{-1/\sigma} = \beta(1+r) C_{t+1}^{-1/\sigma}.$$

In balanced-growth equilibrium, consumption, like productivity, grows at the (gross) rate $1+g$, so

$$1+g=\frac{C_{t+1}}{C_t}=(1+r)^{\sigma}\beta^{\sigma},$$

which tells us the interest rate is

$$r=\frac{(1+g)^{\frac{1}{\sigma}}}{\beta}-1.$$

Now combine this solution with eq. (5),

$$\bar{g}=\theta L-\frac{1}{\alpha}\left[\frac{(1+\bar{g})^{\frac{1}{\sigma}}}{\beta}-1\right],$$

to infer the *equilibrium* rate of growth as the solution to

$$\alpha\beta\bar{g}+(1+\bar{g})^{\frac{1}{\sigma}}=\beta\left(1+\alpha\theta L\right).$$

The solution is illustrated for an arbitrary value of σ by the intersection of the two schedules in figure 7.14 of the Obstfeld-Rogoff book. For example, when $\sigma=1$, we can solve directly and one finds that

$$\bar{g}=\frac{\alpha\beta\theta L-(1-\beta)}{1+\alpha\beta}.$$

Growth is higher for higher L, α, β, σ, and θ. (Why?) One also finds that

$$\bar{r}=\frac{\alpha\left(1+\theta L-\beta\right)}{1+\alpha\beta}.$$

Government policy can certainly affect the economic growth rate in this model. For example, suppose the government imposes a fixed fee τ that new firms have to pay for a license to enter the capital-goods industry. This will increase the sunk cost of entry into the production of new capital goods. The break-even condition, based on eqs. (3) and (4), now becomes

$$\bar{p}_A+\tau=\frac{1+r}{r}\left(\frac{1-\alpha}{\alpha}\right)\left(\frac{\alpha^2}{1+r}\right)^{\frac{1}{1-\alpha}}\bar{L}_Y.$$

Intuitively, as τ rises from 0, $\bar{p}_A$ will fall and $\bar{L}_Y$, will rise. But with $\bar{L}_A=L-\bar{L}_Y$ therefore lower, the pace of productivity growth will be lower as well.

Equilibrium versus Optimal Growth

Due to the presence of monopoly in the system, the equilibrium allocation is not Pareto efficient. Unless there are monopoly profits, producers of capital goods cannot cover their sunk costs, so there is no demand for blueprints, no innovation, and no growth. (Indeed, growth can never even get started.) A second source of inefficiency is the externality in the production of blueprints. An omnipotent central planner, however, could achieve a superior allocation by fiat: researchers would be ordered to produce the optimal flow of blueprints, capital producers to use them to provide the socially optimal level of each capital good. Growth would not simply be maximized in the command allocation, however, because that would require too great a sacrifice of consumption. Instead, there is an optimal trade-off between consumption and R&D. For example, in the case $\sigma = 1$, the planner would use the Lagrangian

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left\{ \log \left[A_t K_t^{\alpha} \left(L - L_{A,t} \right)^{1-\alpha} - A_{t+1} K_{t+1} \right] - \lambda_t \left(A_{t+1} - A_t - \theta A_t L_{A,t} \right) \right\}.$$

It turns out that the optimal growth rate is

$$\bar{g}^* = \beta\theta L - (1-\beta) > \bar{g} = \frac{\alpha\beta\theta L - (1-\beta)}{1+\alpha\beta}.$$

If the government subsidizes capital producers to allow them to produce profitably at social marginal cost, we get a growth rate of

$$\bar{g}^{\text{SUBSIDY}} = \frac{\beta\theta L - (1-\beta)}{1+\beta} < \bar{g}^*.$$

This exceeds the free-market growth rate because the monopoly distortion has been fixed, but falls short of the optimal growth rate because there is no subsidy to make the R&D sector internalize the knowledge externality.

Kremer's (1993) Paper

In a well-known paper, Michael Kremer examined the corollary of the preceding type of model that growth is higher when population is higher. He takes the position that knowledge diffuses across borders, so that the appropriate object of analysis is global population.

Technology growth in Kremer's setup is

$$A_{t+1} - A_t = \theta A_t L_t$$

where L is the global population/workforce. Here, people can have ideas even while producing final consumption goods. In the presence of some fixed factor such as land, output is given by

$$Y_t = A_t L_t^{1-\alpha}.$$

Population growth is endogenous in Kremer's model and given by a Malthusian assumption: population instantaneously rises to the point where per capita output/consumption just equals the minimal level that sustains life. Let us be a bit more generous and allow the minimal living standard to be higher, and to rise over time in a way that (crudely) reflect technology

$$C^{\text{MIN}}\sqrt{A_t} = \frac{Y_t}{L_t}.$$

Combining the last two equations yields

$$A_t = \left(C^{\text{MIN}} L_t^{\alpha}\right)^2.$$

Substituting into $A_{t+1} - A_t = \theta A_t L_t$ leads to

$$L_{t+1}^{2\alpha} - L_t^{2\alpha} = \theta L_t^{1+2\alpha},$$

or

$$\frac{L_{t+1}}{L_t} = (1 + \theta L)^{\frac{1}{2\alpha}}.$$

Population growth, according to this relation, should accelerate over time. Kremer finds support for this prediction on data from 1 million B.C. through 1990. He also finds that between the disappearance of land bridges between the continents and about 1500 (when the Age of Exploration began), the larger continents had faster population growth. Assuming initial populations were proportional to surface area, this prediction too confirms the theory.

Economics 202A

Problem Set #4*

1. *An endogenous growth model based on human capital.* Consider an economy with a fixed labor force. Output per worker is given by

$$y = Ak^{\alpha}(uh)^{1-\alpha}$$

where k is physical capital (per worker), h is human capital (per worker), and $u \in [0, 1]$ is the fraction of the human capital stock allocated to production of output. The rest of the human capital is used to produce new human capital, which depreciates at rate δ:

$$\dot{h} = B(1-u)h - \delta h.$$

Here, A and B are constant. The stocks k and h are predetermined state variables as, therefore, is their ratio,

$$\omega \equiv k/h.$$

The representative household maximizes

$$\int_0^{\infty} v[c(t)]e^{-\theta t} dt$$

subject to the preceding two equations and

$$\dot{k} = y - c - \delta k,$$

where $v(c) = (1-\sigma^{-1})c^{1-\sigma^{-1}}$ and σ is the intertemporal substitution elasticity.

(a) Show via the Maximum Principle that the intertemporal Euler equation for the household's consumption is

$$\frac{\dot{c}}{c} = \sigma \left[\alpha A u^{1-\alpha}\omega^{-(1-\alpha)} - \delta - \theta\right].$$

(b) A second control variable in this optimization problem is u. Define $\chi \equiv c/k$. Show that the Euler equation for u has the form

$$\frac{\dot{u}}{u} = -\chi + Bu + B\left(\frac{1-\alpha}{\alpha}\right).$$

(c) Define $z \equiv Au^{1-\alpha}\omega^{-(1-\alpha)}$. Use the $\dot{c}/c$ and $\dot{k}$ equations above to conclude:

$$\frac{\dot{\chi}}{\chi} = (\alpha\sigma - 1)z + \chi - [\sigma\theta + (\sigma - 1)\delta].$$

(d) Recalling that $\omega = k/h$, show that

$$\frac{\dot{\omega}}{\omega} = z - \chi - B(1-u).$$

(e) Use this last equation and the equation for $\dot{u}/u$, together with the definition of z, to derive:

$$\frac{\dot{z}}{z} = (1-\alpha)\left(\frac{B}{\alpha} - z\right).$$

(f) Suppose we considered the differential equation system consisting of the preceding equations of motion for the three variables z, χ, and u. This (self-contained) system is enough to describe the economy. To see why, note that, in effect, the system is allowing us to track χ, u, and $\omega = u(A/z)^{1/(1-\alpha)}$. But at any time, h and k are given by past investment and education decisions, and so $\omega = k/h$ is also a predetermined state variable. Thus, from the model-implied initial value of $\chi(0) = c(0)/k(0)$ we can infer $c(0)$, along with $u(0)$, and thereby track c, u, h, and k.

In a steady state, there is a constant fraction of labor in manufacturing (u), a constant ratio of consumption to capital (c/k), and a constant ratio of physical to human capital (ω). Find the steady state values $\bar{z}$, $\bar{\chi}$, and $\bar{u}$ from the preceding differential equations [and notice that $\bar{\omega} = \bar{u}(A/\bar{z})^{1/(1-\alpha)}$].

(g) Because y, c, k, and h all rise together over time, we have endogenous growth. Using the consumption Euler equation calculate the steady state growth rate of these variables. What is the intuition behind the solution? [Hint: Think back to the Solow model.]

(h) Linearize the system in z, χ, and u around the steady state of part (f), and calculate its characteristic roots, showing that one is negative and two are positive. Is this what you expected? Why or why not?

*This exercise is based on M. Obstfeld, "Foreign Resource Inflows, Saving, and Growth," in K. Schmidt-Hebbel and Luis Serven (eds.), The Economics of Saving and Growth, Cambridge University Press,1999.