Economics 202A
Lecture #2 Outline (version 1.4)

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I have commented on the ad hoc nature of the saving behavior postulated by Solow. The next model assumes instead that people plan ahead in making saving decisions. One advantage of this assumption is that we can do welfare analysis of economic changes. The model delivers "normative" answers to questions such as, "How much *should* a country save?" In its various forms, the following model has many applications in macroeconomics and public finance beyond the analysis of growth.

The Ramsey-Cass-Koopmans Model in Discrete Time

I will initially develop this model in discrete time. Then I will go to the continuous-time limit to derive a mathematical framework comparable to the Solow model's. This will also serve to illustrate the principles of *optimal control theory*, a very useful tool. There are many other approaches to the derivation, such as the one based on dynamic programming in my notes at http://www.econ.berkeley.edu/~obstfeld/ftp/perplexed/cts4a.pdf. Another possible source is Martin Weitzman's book *Income, Wealth, and the Maximum Principle* (Harvard University Press, 2003).

Assumptions:

- There is a single composite good produced with the constant-returns production function for total output, $Y = F(K, N)$. Here, N is population, which I assume equal to the (fully employed) labor force. (Feel free to add labor-augmenting technical change as an exercise.)
- Population growth is $N_t = (1 + n)N_{t-1}$.
- A "generation" lives for a period t and maximizes

$$U_t = u(c_t) + \beta(1 + n)U_{t+1},$$

where $\beta(1 + n) < 1$ and c_t is the consumption of a representative family member on date t. The idea is that you care about your own consumption and the welfare of your $1 + n$ children.

- Capital depreciates at the rate $\delta \in [0,1]$.

Because $U_t = u(c_t) + \beta(1+n)u(c_{t+1}) + \beta^2(1+n)^2 u(c_{t+2}) + \beta^3(1+n)^3 U_{t+3}$, etc., we may assert that the generation born on date $t = 0$ maximizes

$$U_0 = \sum_{t=0}^{\infty} \beta^t (1+n)^t u(c_t)$$

subject to

$$K_{t+1} - K_t = F(K_t, N_t) - N_t c_t - \delta K_t, \;\; K_t \geq 0, \;\; K_0 \text{ given.}$$

Alternatively, we can express the constraints in the intensive form

$$k_{t+1} = \frac{1}{1+n}\left[f(k_t) + (1-\delta)k_t - c_t\right], \;\; k_t \geq 0, \;\; k_0 \text{ given,}$$

where $k \equiv K/N$, $f(k) = F(k,1)$.

Ramsey looked at the case $n = 0$ and $\beta = 1$. The latter assumption may seem paradoxical from a mathematical point of view (isn't the infinite sum defining U_0 likely to be divergent then?), but a problem set will show how Ramsey handled it.

One simplification is to assume the Inada condition on consumption that $\lim_{c \to 0} u'(c) = \infty$. In that case, we can forget about the interim nonnegativity constraints on the capital stock. We will never optimally get close to zero capital, because the marginal utility of consumption would be very high.

It will be useful first to solve the finite-horizon problem

$$\max_{\{c_t\}} \sum_{t=0}^{T} \beta^t (1+n)^t u(c_t)$$

subject to

$$k_{t+1} = \frac{1}{1+n}\left[f(k_t) + (1-\delta)k_t - c_t\right], \;\; (1+n)k_{T+1} \geq 0, \;\; k_0 \text{ given.}$$

Here, k_{T+1} is the capital left over after consumption in the last period, period T, and it is defined as $k_{T+1} = K_{T+1}/(1+n)N_T$ (since T is the last period of economic activity). We can write the Langrangian for this (Kuhn-Tucker) problem as

$$\sum_{t=0}^{T} \beta^t (1+n)^t \left\{ u(c_t) + \lambda_t \left[f(k_t) + (1-\delta)k_t - c_t - (1+n)k_{t+1} \right] \right\}$$
$$+\beta^T (1+n)^T \eta (1+n) k_{T+1}.$$

The necessary conditions for an optimum are

$$u'(c_t) = \lambda_t; \tag{1}$$

$$-\beta^t (1+n)^{t+1} \lambda_t + \beta^{t+1} (1+n)^{t+1} \lambda_{t+1} \left[f'(k_{t+1}) + 1 - \delta \right] = 0,$$

or

$$\lambda_t = \beta \left[1 + f'(k_{t+1}) - \delta \right] \lambda_{t+1}; \tag{2}$$

and, finally, differentiating with respect to the terminal stock k_{T+1},

$$-\lambda_T \beta^T (1+n)^{T+1} + \beta^T (1+n)^{T+1} \eta = 0,$$

or

$$\eta = \lambda_T.$$

Finally, the Kuhn-Tucker complementary slackness condition can be written as

$$\beta^T (1+n)^T \eta k_{T+1} = \beta^T (1+n)^T \lambda_T k_{T+1} = 0. \tag{3}$$

This implies $k_{T+1} = 0$ because normally, $\lambda_T = u'(c_T) > 0$.

If we combine (2) with (1), we obtain a necessary optimality condition referred to as the *Euler equation* (for capital); we will see it in different forms many times in this course:

$$u'(c_t) = \beta \left[1 + f'(k_{t+1}) - \delta \right] u'(c_{t+1}). \tag{4}$$

What is the intuition? The basic idea is that, if the consumption path is optimal, the initial planner must be indifferent between the two alternatives:

1. Consume a unit of output today, reaping the utility gain $u'(c_t)$.

2. Invest the output in capital instead, but consume the proceeds (including what is left of the initial amount invested) tomorrow. The proceeds are $1 + f'(k_{t+1}) - \delta$, and their marginal utility value from the standpoint of date t is the product of (a) the discount factor $\beta(1+n)$ and (b) the marginal utility reaped by each member of generation $t+1$, $u'(c_{t+1})/(1+n)$. Thus, the marginal gain from this second alternative is $\beta\left[1 + f'(k_{t+1}) - \delta\right] u'(c_{t+1})$, the right-hand side of (4) above.

For the finite horizon problem, this equation completely determines the optimal consumption/accumulation path, together with the initial condition that k_0 is given and the terminal condition that $k_{T+1} = 0$ (for optimality, all capital is eaten when the economy ends).

Formally, if we stare at the two equations

$$u'(c_t) = \beta\left[1 + f'(k_{t+1}) - \delta\right] u'(c_{t+1}), \tag{5}$$

$$k_{t+1} = \frac{1}{1+n}\left[f(k_t) + (1-\delta)k_t - c_t\right],$$

and rewrite them (after some substitution) in the equivalent form

$$c_{t+1} = u'^{-1}\left[\frac{u'(c_t)}{\beta\left(1 + f'\left\{\frac{1}{1+n}\left[f(k_t) + (1-\delta)k_t - c_t\right]\right\} - \delta\right)}\right],$$

$$k_{t+1} = \frac{1}{1+n}\left[f(k_t) + (1-\delta)k_t - c_t\right],$$

where u'^{-1} is the inverse of the function $u'(c)$, then we have a system of two (generally nonlinear) difference equations of the form

$$\begin{aligned} c_{t+1} &= \Phi(c_t, k_t), \\ k_{t+1} &= \Psi(c_t, k_t). \end{aligned} \tag{6}$$

For example, if we have the isoelastic utility function

$$u(c) = \frac{c^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}}, \ \sigma > 0,$$

where σ is the *elasticity of intertemporal substitution*, then $u'(c) = c^{-1/\sigma}$ and $u'^{-1}(x) = x^{-\sigma}$.

In the preceding system (6), k is predetermined by the past history of accumulation, whereas c is a freely-chosen control variable. This type of system is absolutely omnipresent in macroeconomics. In general, there are infinitely many solutions to the two preceding equations – the family of solutions contains two free parameters – but the two boundary conditions that k_0 is given and that $k_{T+1} = 0$ suffice to yield the unique economically-relevant solution. That is, k_0 is a given explicitly and c_0 is given implicitly by the condition that, starting at (c_0, k_0), the equations in (6) land the economy at $k_{T+1} = 0$ in the final period. In contrast, the Solow model could be reduced to a single difference (or differential) equation in k with k_0 providing the one boundary condition needed to pin down the relevant solution path.

Infinite-Horizon Case

The economy's dynamic behavior in the infinite horizon case is still governed by the equations in (5). Also, the predetermined variable k_0 still provides one boundary condition for the system. But with an infinite horizon there clearly is no "terminal" condition on capital in the same simple sense as in a finite-horizon economy.

The relevant terminal condition for the infinite-horizon case, just as in the finite-horizon case, can be derived, however, from eq. (3). Passing to the limit, the latter condition becomes the *transversality condition*,

$$\lim_{T\to\infty} \beta^T (1+n)^T u'(c_T) k_{T+1} = 0. \tag{7}$$

More detailed discussion of the necessity of this condition can be found elsewhere, for example, in the back of Barro and Sala-i-Martin or in the book by Weitzman (*op. cit.*). Important point: While the finite-horizon version of (7) normally implies that $k = 0$ when the economy "ends," (7) itself assumes that there is no end of time and therefore there is no implication that $k \to 0$ asymptotically in the infinite-horizon case. More typically, c and k will both converge to some steady values $\bar{c}$ and $\bar{k}$ in equilibrium, but (7) will hold true nonetheless because $\beta(1+n) < 1$. (As we shall see, however, (7) can hold also when no steady state exists.)

The intuition is also given by David Romer in his text. Imagine a path along which consumption is falling and k is therefore growing very large. Along such a path the product $u'(c)k$ would grow rapidly, probably causing the limit in the last equation to be positive. Such a path could not be optimal, however, because the economy is accumulating excessive hoards of capital, the output of which never gets consumed because it is reinvested instead. It would pay for the economic planner to slightly and permanently increase consumption, an option that is perfectly feasible given the rapid growth in k. Generally, (7) can be applied to rule out initial values c_0 that result in consumption falling over time relative to k. Values of c_0 that are initially too high, and that result in c/k rising asymptotically, generally force k to zero in finite time and are shut off by the nonegativity constraint on capital.

The next section looks at a linear variant of the model and shows how the transversality condition (7) can be applied in practice.

Solving Linear Difference Equation Systems

I simultaneously make the preceding mathematical ideas concrete, while illustrating a critically important solution technique for linear macroeconomic models. Any model into which asset prices and forward-looking expectations enter will have a similar structure (although the dimensionalities of the sets of "state" and "jumping" variables may be bigger; see Blanchard-Kahn on the reading list for a general approach). Here, consumption is the forward-looking jump variable and capital the predetermined state variable. See also Obstfeld and Rogoff, *Foundations of International Macroeconomics*, Supplement C to Chapter 2.

Here we study the infinite-horizon case ($T \to \infty$) in order to illustrate the use of transversality arguments to determine appropriate boundary conditions.

I make two critical simplifications: $u(c) = \ln c$ (corresponding to an intertemporal substitution elasticity $\sigma = 1$; why?), and $f(k) = Ak$ (so that capital is the only productive factor). Define

$$\rho \equiv 1 + A - \delta.$$

Then the equations in (6) take the simple linear form

$$\begin{aligned} c_{t+1} &= \beta \rho c_t, \\ k_{t+1} &= -\frac{1}{1+n} c_t + \frac{\rho}{1+n} k_t. \end{aligned}$$

In matrix notation, this is

$$\begin{bmatrix} c_{t+1} \\ k_{t+1} \end{bmatrix} = \begin{bmatrix} \beta\rho & 0 \\ -\frac{1}{1+n} & \frac{\rho}{1+n} \end{bmatrix} \begin{bmatrix} c_t \\ k_t \end{bmatrix}. \tag{8}$$

How does one solve such a system?

[*Important note*: Because there are constant marginal returns to capital, not diminishing returns as we have assumed heretofore, this model has no balanced growth path in the sense of Solow. Instead, per capita consumption grows steadily over time. This is an early (and rudimentary) example of an *endogenous growth* model. Growth is endogenous here because it is not driven by any exogenous process of technical change, as in Solow's model; it comes instead from the intrinsic economic mechanisms in the model. That special nature of this model gives its solution a somewhat different form from that of more standard rational expectations models, as I shall note below.]

We know that for a simple univariate linear difference equation of the form $y_{t+1} = \zeta y_t$, the general solution has the form $y_t = a\zeta^t$, where a is an arbitrary constant. (In a specific application, some particular boundary condition on y would allow us to pin a down uniquely.) We therefore proceed by diagonalizing the matrix in the last expression, applying the simple univariate solution, and then reversing the diagonalization process to solve for c and k.

To be precise, express the matrix system (8) in vector notation as

$$y_{t+1} = My_t,$$

where M is the 2×2 matrix displayed above. Suppose we can find an invertible 2×2 matrix X such that $X^{-1}MX = \Gamma$ is *diagonal*. Then we would have

$$X^{-1}y_{t+1} = X^{-1}MX\left(X^{-1}y_t\right),$$

or, defining $\tilde{y} \equiv X^{-1}y$,

$$\tilde{y}_{t+1} = X^{-1}MX\tilde{y}_t = \Gamma\tilde{y}_t.$$

This (vector) difference equation is easy to solve as

$$\tilde{y}_{t+1} = \Gamma^{t+1}\tilde{y}_0,$$

where $\tilde{y}_0$ is an initial condition for the vector $\tilde{y}$. We can then retrieve the solution for y itself via the linear transformation

$$y_{t+1} = X\Gamma^{t+1}\tilde{y}_0 = X\Gamma^{t+1}X^{-1}y_0.$$

Finding a matrix X such that $MX = X\Gamma$ is standard linear algebra. There are many such matrices, and all we need is one – so let's restrict our search to X with $x_{21} = x_{22} = 1$. Let

$$\Gamma = \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix}$$

and write the preceding condition defining X as

$$M \begin{bmatrix} x_{11} & x_{12} \\ 1 & 1 \end{bmatrix} = X\Gamma = \begin{bmatrix} \gamma_1 x_{11} & \gamma_2 x_{12} \\ \gamma_1 & \gamma_2 \end{bmatrix}.$$

Thus, the vectors $\begin{bmatrix} x_{1i} \\ 1 \end{bmatrix}$ have the property that $M \begin{bmatrix} x_{1i} \\ 1 \end{bmatrix} = \gamma_i \begin{bmatrix} x_{1i} \\ 1 \end{bmatrix}$. (They are *eigenvectors* and the two γ_i are *eigenvalues*.)

Because the mapping $M - \gamma_i I$ (where I is the 2×2 identity matrix) maps both the nonzero vector $\begin{bmatrix} x_{1i} \\ 1 \end{bmatrix}$ and the zero vector to the zero vector, the matrix $M - \gamma_i I$ is noninvertible (i.e., singular) and has a zero determinant:

$$\det(M - \gamma_i I) = 0.$$

This equation tells us that we can find the eigenvalues by solving the equation

$$\det \begin{bmatrix} \beta\rho - \gamma_i & 0 \\ -\frac{1}{1+n} & \frac{\rho}{1+n} - \gamma_i \end{bmatrix} = 0,$$

which is equivalent to

$$\gamma_i^2 - \rho\left(\beta + \frac{1}{1+n}\right)\gamma_i + \frac{\beta\rho^2}{1+n} = 0.$$

It is easy to check that the two roots of this quadratic are

$$\{\gamma_1, \gamma_2\} = \left\{\rho\beta, \frac{\rho}{1+n}\right\}.$$

The eigenvectors are found by using

$$\begin{bmatrix} \beta\rho & 0 \\ -\frac{1}{1+n} & \frac{\rho}{1+n} \end{bmatrix} \begin{bmatrix} x_{1i} \\ 1 \end{bmatrix} = \gamma_i \begin{bmatrix} x_{1i} \\ 1 \end{bmatrix}$$

or solving either of the two equations

$$\beta\rho x_{1i} = \gamma_i x_{1i}$$

and

$$-\frac{1}{1+n}x_{1i}+\frac{\rho}{1+n}=\gamma_i.$$

For $\gamma_1=\rho\beta$, the first of the last two equations above is uninformative (it holds for any x_{11}), but the second requires that

$$x_{11}=\rho-(1+n)\rho\beta.$$

For $\gamma_2=\rho/(1+n)$, however, the first of the two preceding equations defining eigenvectors holds only for $x_{12}=0$, whereas the second equation reduces to the true relationship $\gamma_2=\rho/(1+n)$ in that case. As a result, a viable matrix X is given by

$$X=\begin{bmatrix}\rho\left[1-(1+n)\beta\right] & 0\\ 1 & 1\end{bmatrix}.$$

We are almost there. Recall that we now want to transform the original system by X^{-1}, where

$$\begin{aligned}X^{-1} &= \frac{1}{\rho\left[1-(1+n)\beta\right]}\begin{bmatrix}1 & 0\\ -1 & \rho\left[1-(1+n)\beta\right]\end{bmatrix}\\ &= \begin{bmatrix}\frac{1}{\rho[1-(1+n)\beta]} & 0\\ -\frac{1}{\rho[1-(1+n)\beta]} & 1\end{bmatrix}.\end{aligned}$$

Thus,

$$X^{-1}\begin{bmatrix}c_{t+1}\\ k_{t+1}\end{bmatrix}=\begin{bmatrix}\tilde{c}_{t+1}\\ \tilde{k}_{t+1}\end{bmatrix}=\Gamma\begin{bmatrix}\tilde{c}_t\\ \tilde{k}_t\end{bmatrix},$$

giving us

$$\begin{aligned}\tilde{c}_{t+1} &= (\rho\beta)^{t+1}\tilde{c}_0,\\ \tilde{k}_{t+1} &= \left(\frac{\rho}{1+n}\right)^{t+1}\tilde{k}_0.\end{aligned}$$

Reversing the X^{-1} transformation of (c,k), we finally get the solution

$$\begin{aligned}\begin{bmatrix}c_{t+1}\\ k_{t+1}\end{bmatrix} &= \begin{bmatrix}\rho\left[1-(1+n)\beta\right] & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}(\rho\beta)^{t+1}\tilde{c}_0\\ \left(\frac{\rho}{1+n}\right)^{t+1}\tilde{k}_0\end{bmatrix} \qquad (9)\\ &= \begin{bmatrix}\rho\left[1-(1+n)\beta\right](\rho\beta)^{t+1}\frac{1}{\rho[1-(1+n)\beta]}c_0\\ (\rho\beta)^{t+1}\frac{1}{\rho[1-(1+n)\beta]}c_0+\left(\frac{\rho}{1+n}\right)^{t+1}\left\{k_0-\frac{1}{\rho[1-(1+n)\beta]}c_0\right\}\end{bmatrix}\\ &= \begin{bmatrix}(\rho\beta)^{t+1}c_0\\ (\rho\beta)^{t+1}\dfrac{c_0}{\rho\left[1-(1+n)\beta\right]}+\left(\dfrac{\rho}{1+n}\right)^{t+1}\left\{k_0-\dfrac{c_0}{\rho\left[1-(1+n)\beta\right]}\right\}.\end{bmatrix}\end{aligned}$$

The solution is predicated on two initial conditions, c_0 and k_0. Because the capital stock is predetermined, k_0 is given by the past history of saving. What about c_0? A useful way to think about the appropriate initial condition is to calculate the ratio of capital to consumption, which is

$$\frac{k_t}{c_t} = \frac{1}{\rho\left[1-(1+n)\beta\right]} + \left[\frac{1}{\beta(1+n)}\right]^t \left\{\frac{k_0}{c_0} - \frac{1}{\rho\left[1-(1+n)\beta\right]}\right\}.$$

Recall our prior assumption that $(1+n)\beta < 1$; this implies that the term $\left[\frac{1}{\beta(1+n)}\right]$ in the last expression exceeds 1, so that $\left[\frac{1}{\beta(1+n)}\right]^t$ explodes as $t \to \infty$. Notice that if

$$\frac{k_0}{c_0} > \frac{1}{\rho\left[1-(1+n)\beta\right]},$$

then the ratio of capital to consumption will rise without limit. This cannot possibly be optimal: you could raise utility by a tiny increase in c_0, which could feasibly be maintained forever. Alternatively, think about the transversality condition (7) (and recall that here, $u'(c) = 1/c$ due to log utility). Because

$$\lim_{t\to\infty}(1+n)^T\beta^T\frac{k_T}{c_T} = \left\{\frac{k_0}{c_0} - \frac{1}{\rho\left[1-(1+n)\beta\right]}\right\} > 0,$$

condition (7) is violated. Of course, if $\left\{\frac{k_0}{c_0} - \frac{1}{\rho\left[1-(1+n)\beta\right]}\right\} < 0$ it is violated as well. In that case, the dynamic equations of the model predict that k/c must eventually become negative, which is infeasible. Therefore, the only initial condition of the model consistent with optimality is

$$c_0 = \rho\left[1-(1+n)\beta\right]k_0.$$

Observe that, under this initial condition, the consumption-capital ratio remains constant forever, with consumption and capital alike growing at the (gross) rate $\rho\beta$. (This is the "endogenous growth" prediction of the model.) We know this by plugging the appropriate initial conditions into (9) to get:

$$c_t = \rho\left[1-(1+n)\beta\right](\rho\beta)^t k_0, \;\; k_t = (\rho\beta)^t k_0 \tag{10}$$

There is a useful interpretation of the consumption function. In any period, total available resources are ρk – the capital stock plus the output it generates, net of depreciation. With log utility the optimal policy is to consume a fraction $1-(1+n)\beta$ of ρk, where $(1+n)\beta$ is the discount factor for future utility.

Meta-digression on the solution: As noted, this endogenous growth model has characteristics different from most of the models you will encounter in macroeconomics. In particular, there is no possibility for a steady state solution for c and k – only the ratio c/k is constant.

A dynamic system has one characteristic root associated with each endogenous variable. In more standard discrete-time models, each jumping variable is associated with a root (of modulus) greater than 1, while each predetermined variable is associated with a root (of modulus) less than 1.[1] By imposing the transversality condition (or something akin to it) to derive a unique solution, we generally rule out certain types of explosive behavior by zeroing out the influence of the large roots on the system's intrinsic dynamics.

Here, both roots $\left\{\rho\beta, \dfrac{\rho}{1+n}\right\}$ could well be above 1. Because $\beta(1+n) < 1$, $\rho/(1+n)$ is the larger of these two roots. If you look at the specific solution (10), however, you will see that the transversality condition has been used to eliminate the influence of the larger root on the system's intrinsic dynamics, which instead are driven entirely by the smaller root $\rho\beta$. It is in this sense that the present system's solution is analogous to the solutions for more standard models discussed in the paper by Blanchard and Kahn.

[1] Some of the roots could be complex. These give rise to oscillatory behavior.

Economics 202A, Problem Set 2

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1. (A linearized dynamic model) Consider a case with no population growth and a 100% per period capital depreciation rate, so that the planner maximizes $\sum_{t=0}^{\infty}\beta^t u(c_t)$ subject to $k_{t+1} = f(k_t) - c_t$. Let $u(c) = \log(c)$, $f(k) = k^{\alpha}$, $\alpha < 1$.

(a) Show that the dynamics along the model's optimal path are described by the pair of nonlinear equations:

$$c_{t+1} - c_t = \alpha\beta(k_t^{\alpha} - c_t)^{\alpha-1}c_t - c_t, \quad k_{t+1} - k_t = k_t^{\alpha} - c_t - k_t. \qquad (*)$$

(b) Compute the steady-state values $\bar{c}$ and $\bar{k}$ as functions of α and β. (In an endogenous-growth context you might want to return to this problem and think hard about what happens as $\alpha \to 1$.)

(c) The multivariate version of Taylor's theorem states that if the function $g(x,y)$ is smooth enough, its linear approximation near $(\bar{x},\bar{y})$ is

$$g(x,y) \approx g(\bar{x},\bar{y}) + \frac{\partial g}{\partial x}(\bar{x},\bar{y})(x - \bar{x}) + \frac{\partial g}{\partial y}(\bar{x},\bar{y})(y - \bar{y}).$$

Use this result to show that the system (*) you derived in part (a) has the matrix linear approximation

$$\begin{bmatrix} c_{t+1} - \bar{c} \\ k_{t+1} - \bar{k} \end{bmatrix} = \begin{bmatrix} \alpha + (1-\alpha)/\alpha\beta & -(1-\alpha)(1-\alpha\beta)/\alpha\beta^2 \\ -1 & 1/\beta \end{bmatrix} \begin{bmatrix} c_t - \bar{c} \\ k_t - \bar{k} \end{bmatrix}. \qquad (**)$$

near its steady state.

(d) Show that the characteristic roots of the matrix M in (**) (that is, the solutions γ to the equation $\det(M - \gamma I) = 0$, where I is the identity matrix) are $1/\alpha\beta$ and α. Define the matrix $\Gamma = \begin{bmatrix} 1/\alpha\beta & 0 \\ 0 & \alpha \end{bmatrix}$.

(e) Find any 2 × 2 matrix X such that $MX = X\Gamma$. [Hint: Columns of X are called eigenvectors belonging to $1/\alpha\beta$ and α; they are rays rather than well-defined vectors but you can tie them down by assuming that $x_{21} = x_{22} = 1$, then solving for x_{11} and x_{12} only.]

(f) Why do we need such an X? Define the transformed variables c′, k′ by $\begin{bmatrix} c' \\ k' \end{bmatrix} = X^{-1}\begin{bmatrix} c \\ k \end{bmatrix}$. Show that multiplying (**) through by X^{-1} gives the equation

$$X^{-1}\begin{bmatrix} c_{t+1} - \bar{c} \\ k_{t+1} - \bar{k} \end{bmatrix} = \begin{bmatrix} c'_{t+1} - \bar{c}' \\ k'_{t+1} - \bar{k}' \end{bmatrix} = X^{-1}MXX^{-1}\begin{bmatrix} c_t - \bar{c} \\ k_t - \bar{k} \end{bmatrix} = \Gamma\begin{bmatrix} c'_t - \bar{c}' \\ k'_t - \bar{k}' \end{bmatrix}. \qquad (\dagger)$$

Note with satisfaction that Γ (conveniently and not accidentally) is diagonal.

(g) Show that all solutions to system (†) take the form:

$$c'_{t+1} - \bar{c}' = (1/\alpha\beta)^{t+1}(c'_0 - \bar{c}'), \qquad k'_{t+1} - \bar{k}' = \alpha^{t+1}(k'_0 - \bar{k}'),$$

where t = 0 is an initial date.

(h) Now we want to use these simple solutions to retrieve c and k. Find $c_{t+1} - \bar{c}$ and $k_{t+1} - \bar{k}$ by doing the inverse transformation:

$$X\begin{bmatrix} c' \\ k' \end{bmatrix} = XX^{-1}\begin{bmatrix} c \\ k \end{bmatrix} = \begin{bmatrix} c \\ k \end{bmatrix}.$$

Solve for c_0 by showing that only if $c_0 = \bar{c} + \left(\frac{1}{\beta} - \alpha\right)(k_0 - \bar{k})$ will the system reach its steady state. Solve for $c_t - \bar{c}$ and $k_t - \bar{k}$ along the saddlepath. Show that the planner should set consumption (near the steady state) by the rule

$$c_t = \frac{(1-\alpha\beta)}{\beta} k_t + (\alpha\beta)^{\frac{\alpha}{1-\alpha}}(1-\alpha\beta)(1-\alpha).$$

2. (Zero discounting) In his famous 1928 *Economic Journal* article on optimal saving, Frank P. Ramsey argued that it is morally indefensible to discount the welfare of future generations. He therefore argued that a benevolent economic planner should:

$$\text{maximize } V = \int_0^\infty u[c(t)]dt$$

$$\text{subject to } \dot{k}(t) = f[k(t)] - c(t), \; k(0) \text{ given.}$$

(Ramsey assumed zero population growth.) You can see right away the problem with this formuation: <u>any</u> path that approaches a constant steady-state consumption level will yield an <u>infinite</u> value of V. Thus, it is not clear how to compare such paths and identify one as "optimal."

Ramsey finessed the problem in the following way. He defined $\bar{c}$ to be the "bliss" or maximal steady-state consumption level [the existence of which presupposes that f(k) either eventually becomes decreasing in k or asymptotes to a finite maximum as k goes to ∞]. He then redefined his problem as that of <u>minimizing</u> $\int_0^\infty \{u(\bar{c}) - u[c(t)]\}dt$, society's cumulative distance from "bliss", subject to the above constraints. Note that this integral can be finite if $c(t) \to \bar{c}$ as $t \to \infty$ (and if it isn't, it's not the optimum we seek in any case).

(a) Use the Maximum Principle to derive necessary conditions for a solution to the Ramsey problem. (You can assume a depreciation rate of 0 for capital. The resulting Euler condition is sometimes called the Keynes-Ramsey rule because J. M. Keynes, a friend of Ramsey's and editor of the *Economic Journal*, helped him to interpret it intuitively.) Show the economy indeed should converge to "bliss" (also known as the "golden rule" in growth theory.) Interpret the model's intertemporal Euler condition. You can do so by adressing the following question: Suppose the economy starts with $k(0) < \bar{k}$. Since Ramsey believed in intergenerational equality, why isn't it optimal in his view for each generation simply to consume f[k(0)]?

(b) Let $\{c^*(t)\}_{t=0}^\infty$ denote the Ramsey consumption path starting from an initial capital stock k(0), and let $\{c(t)\}_{t=0}^\infty$ be any other consumption path. Show that the Ramsey path *overtakes* any other feasible consumption path starting from k(0), in the following sense: there exists a finite time $\mathcal{T}$ such that for all $T > \mathcal{T}$, $\int_0^T u[c^*(t)]dt > \int_0^T u[c(t)]dt$.

(c) Ramsey states the Keynes-Ramsey rule as: "*rate of saving multiplied by marginal utility of consumption should always equal bliss minus actual rate of utility enjoyed.*" [By "bliss" he meant $u(\bar{c})$.] Can you derive this rule?