Economics 202A
Lecture Outline # 8 (version 1.4)

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Stock Prices as Present Values

The most basic theory of the stock market is that a stock's price is the present value of expected future dividends.

Suppose the real interest rate is r, and is constant. Suppose the stocks real dividend in period t is d_t and the stock's *ex dividend* real price (i.e., in terms of output, or more generally, in terms of the CPI basket), is q_t.

Then in a risk-neutral world, we would have the arbitrage condition

$$1 + r = \mathrm{E}_t \left\{ \frac{d_{t+1} + q_{t+1}}{q_t} \right\}, \tag{1}$$

which equates the gross return on bonds to that on stocks (dividends + capital gains). This works for a time-dependent interest rate r_t as well — do that case as an exercise.

To see how the preceding return relationship translates into a theory of stock pricing, write

$$\begin{aligned} q_t &= \mathrm{E}_t \left\{ \frac{d_{t+1} + q_{t+1}}{1+r} \right\} \\ &= \mathrm{E}_t \left\{ \frac{d_{t+1}}{1+r} \right\} + \mathrm{E}_t \left\{ \frac{q_{t+1}}{1+r} \right\} \\ &= \mathrm{E}_t \left\{ \frac{d_{t+1}}{1+r} \right\} + \mathrm{E}_t \left\{ \frac{1}{1+r} \mathrm{E}_{t+1} \left\{ \frac{d_{t+2} + q_{t+2}}{1+r} \right\} \right\} \\ &= \mathrm{E}_t \left\{ \frac{d_{t+1}}{1+r} \right\} + \mathrm{E}_t \left\{ \frac{d_{t+2}}{(1+r)^2} \right\} + \mathrm{E}_t \left\{ \frac{q_{t+2}}{(1+r)^2} \right\}. \end{aligned}$$

Here, I have used the law of iterated conditional expectations, $\mathrm{E}_t \{\mathrm{E}_{t+1} x_{t+2}\} = \mathrm{E}_t \{x_{t+2}\}$.

One can continue the iterative substitution procedure above indefinitely, successively substituting the versions of eq. (1) for dates $t+2$, $t+3$, etc.

The result is

$$q_t = \sum_{i=1}^{\infty} \mathrm{E}_t \left\{ \frac{d_{t+i}}{(1+r)^i} \right\} + \lim_{i \to \infty} \mathrm{E}_t \left\{ \frac{q_{t+i}}{(1+r)^i} \right\}.$$

What to make of the term $\lim_{i \to \infty} \mathrm{E}_t \left\{ \frac{q_{t+i}}{(1+r)^i} \right\}$? This term represents a potential *speculative bubble* (of one particular "rational" kind) in the stock price: it captures the idea of a self-fulfilling frenzy in the asset price. More on this later; for now let's assume there is no bubble. In that case

$$q_t = \mathrm{E}_t \left\{ \sum_{i=1}^{\infty} \frac{d_{t+i}}{(1+r)^i} \right\}, \tag{2}$$

and the stock's price is the expected present value of future dividends.

An important implication of this formula is that changes in stock prices reflect *news*.

Suppose that, within a particular trading instant, people change their expected dividend stream to be $\mathrm{E}'_t \{d_{t+1}\}$. Then the stock price will jump by the amount

$$q'_t - q_t = \mathrm{E}'_t \left\{ \sum_{i=1}^{\infty} \frac{d_{t+i}}{(1+r)^i} \right\} - \mathrm{E}_t \left\{ \sum_{i=1}^{\infty} \frac{d_{t+i}}{(1+r)^i} \right\},.$$

where this change is uncorrelated with any information available before the revision in market expectations. This is the basic idea of the "random walk" theory of stock prices, or, more broadly, the "efficient markets" view.

As another application, consider the behavior of the stock price from period to period. We have

$$q_{t+1} - q_t = \mathrm{E}_{t+1} \left\{ \sum_{i=1}^{\infty} \frac{d_{t+1+i}}{(1+r)^i} \right\} - \mathrm{E}_t \left\{ \sum_{i=1}^{\infty} \frac{d_{t+i}}{(1+r)^i} \right\}.$$

Let dividends follow the AR(1) process

$$d_{t+1} = \rho d_t + \varepsilon_{t+1},$$

where $\mathrm{E}_t \varepsilon_{t+1} = 0$. Then

$$q_t = \frac{\rho}{1+r-\rho} d_t$$

and

$$\begin{aligned} q_{t+1} - q_t &= \frac{\rho}{1+r-\rho}\left(d_{t+1} - d_t\right) \\ &= \frac{\rho}{1+r-\rho}\left[(\rho-1)d_t + \varepsilon_{t+1}\right]. \end{aligned}$$

Changes in stock prices are proportional to changes in dividends (as in Shiller's excess volatility tests). Also, for ρ near 1, or for a very small time interval, the change in the stock price is essentially proportional to the "news" ε_{t+1} — the innovation in dividends.

We get at the essence of the "efficient markets" hypothesis by examining the ex post *excess return*

$$e_{t+1} = \frac{d_{t+1} + q_{t+1}}{q_t} - 1 - r.$$

Our arbitrage condition guarantees that this is uncorrelated with date t information. In our particular AR(1) example,

$$\begin{aligned} \frac{d_{t+1} + q_{t+1}}{q_t} - 1 - r &= \frac{d_{t+1} + \frac{\rho}{1+r-\rho} d_{t+1}}{\frac{\rho}{1+r-\rho} d_t} - 1 - r \\ &= \frac{(1+r)d_{t+1}}{\rho d_t} - 1 - r \\ &= \frac{(1+r)\left(\rho d_t + \varepsilon_{t+1}\right)}{\rho d_t} - 1 - r \\ &= \frac{\varepsilon_{t+1}}{\rho d_t}. \end{aligned}$$

For any random variable x_t realized as of date t, $\mathrm{E}_t\left\{\frac{\varepsilon_{t+1}}{\rho d_t} x_t\right\} = \frac{x_t}{\rho d_t}\mathrm{E}_t \varepsilon_{t+1} = 0$. The excess return is *unpredictable*.

Note: Even if there is a "rational bubble" in the stock price the preceding implication of unpredictable excess returns will hold. That is because the result follows entirely from eq. (1), rather than from eq. (2).

Summers's Critique on the Interpretation of Efficiency Tests

Some financial economists argued that if one fails to find lagged variables helping to predict excess returns e_t, one can infer that the PDV formula (2) for a stock's price is valid: stocks are priced according to their fundamentals. Larry Summers offers a persuasive critique of this inference in his paper "Does the Stock Market Rationally Reflect Fundamental Values?" on the reading list.

Let q_t^* (temporarily, for this section) denote the PDV price given in equation (2) and imagine that, perhaps do to "fads" in investment preferences or the like, the *actual* stock price q_t is given by

$$q_t = q_t^* \mathrm{e}^{u_t},$$

where the log discrepancy u_t follows an autoregressive process

$$u_t = \alpha u_{t-1} + v_t, \qquad |\alpha| \leq 1,$$

where the innovation v_t is uncorrelated with all economic variables at all leads and lags. (It is a pure "sunspot.") In this alternative model, stock prices can differ from fundamental values due to a slow moving pricing error that can be expected to diminish over time if $|\alpha| < 1$. The question Summers asks is: will standard tests of excess return predictability disclose the presence of this — possibly large — pricing error? His answer is no.

Let's see why. Define the efficient excess return as

$$e_{t+1} = \frac{d_{t+1} + q_{t+1}^*}{q_t^*} - 1 - r$$

and, following Summers, define the actual excess return as

$$z_{t+1} = \frac{d_{t+1} + q_{t+1}}{q_t} - 1 - r$$

Let's adopt the approximations $\frac{q_{t+1} - q_t}{q_t} \approx \log q_{t+1} - \log q_t$ and $\mathrm{e}^{-u_t} \approx 1 - u_t$. (The latter is not going to be a great approximation unless u_t is relatively small, but I am getting closer to the right answer than Summers does. He assumes that $\frac{d_{t+1}}{q_t^*} \approx \frac{d_{t+1}}{q_t}$, which amounts to the very bad approximation $\mathrm{e}^{-u_t} \approx 1$. I worry about Summers's approximation because it is only good when $q_t \approx q_t^*$, whereas the whole point of this exercise is to argue that the two q's can diverge widely.) Then we may write

$$\begin{aligned} z_{t+1} &\approx \log q_{t+1} - \log q_t + \frac{d_{t+1}}{q_t^*}\mathrm{e}^{-u_t} - r \\ &\approx \log q_{t+1}^* - \log q_t^* + u_{t+1} - u_t + \frac{d_{t+1}}{q_t^*} - \frac{d_{t+1}}{q_t^*}u_t - r \\ &\approx e_{t+1} + u_{t+1} - u_t - \frac{d_{t+1}}{q_t^*}u_t. \end{aligned}$$

To finish up, imagine that dividends follow the AR(1) process $d_{t+1} = \rho d_t + \varepsilon_{t+1}$. To make life easier, let us take $\rho = 1$. As per our earlier result, we have $q_t^* = d_t\,/r$ and so

$$\frac{d_{t+1}}{q_t^*}u_t = \frac{\rho d_t}{q_t^*}u_t + \frac{\varepsilon_{t+1}}{q_t^*}u_t \approx r u_t,$$

assuming that $\frac{\varepsilon_{t+1}}{q_t^*}u_t$ is small. So

$$\begin{aligned} z_{t+1} &\approx e_{t+1} + u_{t+1} - u_t - r u_t \\ &= e_{t+1} + v_{t+1} + (\alpha - r - 1)\,u_t. \end{aligned}$$

Using this approximation, and the fact that $\sigma_u^2 = \sigma_v^2/(1-\alpha^2)$, we find that the variance of z is

$$\begin{aligned} \sigma_z^2 &= \sigma_e^2 + \sigma_u^2(1-\alpha^2) + (\alpha - r - 1)^2\,\sigma_u^2 \\ &= \sigma_e^2 + \left[2(1+r)(1-\alpha) + r^2\right]\sigma_u^2. \end{aligned}$$

Since the interest rate r is the rate from month to month, it is small in magnitude and this formula is close to Summers's. Let ρ_1 be the first lagged autocorrelation of z, $\rho_1 \equiv \mathrm{Corr}(z_{t+1}, z_t)$. It is proportional to the covariance

$$\begin{aligned} & \mathrm{E}\left[e_{t+1} + v_{t+1} + (\alpha - r - 1)\,u_t\right]\left[e_t + v_t + (\alpha - r - 1)\,u_{t-1}\right] \\ =\; & \mathrm{E}\left[e_{t+1} + (\alpha - r - 1)\,\alpha u_{t-1} + (\alpha - r - 1)\,v_t\right]\left[e_t + v_t + (\alpha - r - 1)\,u_{t-1}\right] \\ =\; & \left[(\alpha - r - 1)\,(1-\alpha^2) + \alpha\,(\alpha - r - 1)^2\right]\sigma_u^2 = [\alpha - (1+r)][1 - \alpha(1+r)]\sigma_u^2. \end{aligned}$$

Thus

$$\rho_1 = -\frac{[1 + r - \alpha][1 - \alpha(1+r)]\sigma_u^2}{\sigma_e^2 + \left[2(1+r)(1-\alpha) + r^2\right]\sigma_u^2},$$

which is less than 0 unless α is very close to 1. When $r = 0$, this is the same as in the Summers paper.

How big is the autocorrelation likely to be? Summers suggests taking $\alpha = 0.98$ for monthly data. In this case the fraction of a u innovation that has not decayed after three years is $0.98^{36} = 0.483$. That is, the half-life of a "fad" is about 3 years. Summers also suggests that we take $\sigma_e^2 = 0.001$ (making the monthly standard deviation of returns about 3.2 percent). Finally, he looks at the case $\sigma_u^2 = .08$, meaning that the market is making valuation errors with a standard deviation of nearly 30 percent. Finally, I add the assumption that $r = 0.00325$ or 0.325 per cent per month, giving an annual real interest rate of about 4 percent. Then we find that

$$\rho_1 = -0.00743,$$

which is slightly smaller than Summers estimate of -0.008. Of course, higher-order autocorrelations are even smaller. Summers's point is that it would take thousands of years of monthly data to reliably detect such a small autocorrelation in excess returns — even though q deviates persistently from q^* by large amounts. The intuition for this example is that the deviation $\log q^* - \log q$ is so persistent that it can barely be detected by looking at the autocorrelation in returns. The overall deviation u has a high variance, but its innovation v need not, so the example is not terribly far off from adding a constant to the stock price. In contrast, a more variable higher-frequency noise would be easier to detect.

Risk and Equity Pricing

When people are risk averse, the relevant arbitrage condition between bonds and equities is more complicated. The model of Robert E. Lucas, Jr. ("Asset Prices in an Exchange Economy," *Econometrica*, November 1978) deals with this case. The starting point is the Euler equation for the stock, which can be written (for a representative agent) as

$$q_t = \mathrm{E}_t \left\{ \frac{\beta u'(c_{t+1})}{u'(c_t)} \left(d_{t+1} + q_{t+1}\right) \right\}. \tag{3}$$

Notice that the "risk neutral" formula we used before is different. Because the bond Euler equation states that

$$\frac{1}{1+r} = \mathrm{E}_t\left\{\frac{\beta u'(c_{t+1})}{u'(c_t)}\right\},$$

the preceding risk-neutral hypothesis that $q_t = \mathrm{E}_t\left\{\frac{d_{t+1}+q_{t+1}}{1+r}\right\}$ would, in the Lucas model, imply the invalid relationship

$$q_t = \mathrm{E}_t\left\{\mathrm{E}_t\left\{\frac{\beta u'(c_{t+1})}{u'(c_t)}\right\}(d_{t+1}+q_{t+1})\right\},$$

which is generally the same as eq. (3) if the marginal utility of consumption is constant (no risk aversion) but not otherwise. By using eq. (3), we are also allowing for non-constant real interest rates.

To see how this case differs from the risk-neutral, let us again write the excess return on the equity as

$$e_{t+1} = \frac{d_{t+1}+q_{t+1}}{q_t} - (1+r)$$

and express (3) as

$$1 = \mathrm{E}_t\left\{\frac{\beta u'(c_{t+1})}{u'(c_t)}(e_{t+1}+1+r)\right\},$$

or as

$$0 = \mathrm{E}_t\left\{\frac{\beta u'(c_{t+1})}{u'(c_t)}e_{t+1}\right\},$$

where we have recalled that

$$\mathrm{E}_t\left\{\frac{\beta u'(c_{t+1})}{u'(c_t)}\right\} = \frac{1}{1+r}.$$

An equivalent expression is

$$\begin{aligned} 0 &= \mathrm{E}_t\left\{\frac{\beta u'(c_{t+1})}{u'(c_t)}\right\}\mathrm{E}_t\{e_{t+1}\} + \mathrm{Cov}_t\left\{\frac{\beta u'(c_{t+1})}{u'(c_t)}, e_{t+1}\right\} \Longleftrightarrow \\ 0 &= \frac{\mathrm{E}_t\{e_{t+1}\}}{1+r} + \mathrm{Cov}_t\left\{\frac{\beta u'(c_{t+1})}{u'(c_t)}, e_{t+1}\right\}. \end{aligned}$$

Thus we find that

$$\frac{\mathrm{E}_t\left\{e_{t+1}\right\}}{1+r} = -\mathrm{Cov}_t\left\{\frac{\beta u'(c_{t+1})}{u'(c_t)}, e_{t+1}\right\}. \tag{4}$$

The expected excess return (in terms of today's consumption) equals minus the covariance between the excess return and the (ex post) marginal rate of substitution of present for future consumption. This formula also implies that excess returns can be predictable – in principle, by any information in the information set underlying the conditional covariance in (4).

How should one interpret the fundamental relationship (4)? Imagine that the covariance in the equation is positive. Since $u''(c) < 0$, this means that the excess return tends to be high when consumption is low (i.e., when the marginal utility of consumption is high). In this case the stock provides good *consumption insurance* because it tends to do well when other sources of income are underperforming. So the expected excess return will be negative — the expected return is less than the risk-free rate, because the asset reduces the risk of the overall portfolio. Conversely, if the covariance is negative, we have an asset whose payoff is high when consumption is high. This asset does not help insure against consumption risk, so its expected return must offer a (positive) *risk premium* over the risk-free rate. Note that the relevant concept of risk is *not* the variance of the return; it is the covariance with consumption. That is why this model is often called the consumption-based capital asset pricing model (CCAPM).

A useful approximation can be derived as follows. Assume CRRA preferences and take the second-order Taylor approximation around the point $e_t = 0$, $c_{t+1}/c_t = 1$:

$$\left(\frac{c_{t+1}}{c_t}\right)^{-R} e_{t+1} \approx e_{t+1} - R\left(\frac{c_{t+1}}{c_t} - 1\right) e_{t+1}.$$

Then we may estimate

$$\begin{aligned} 0 &= \mathrm{E}_t\left\{\frac{\beta c_{t+1}^{-R}}{c_t^{-R}} e_{t+1}\right\} \\ &\approx \mathrm{E}_t\left\{\beta e_{t+1} - \beta R\left(\frac{c_{t+1}}{c_t} - 1\right) e_{t+1}\right\}. \end{aligned}$$

This implies that

$$\mathrm{E}_t\left\{e_{t+1}\right\} = R\mathrm{Cov}_t\left\{e_{t+1}, \frac{c_{t+1}}{c_t} - 1\right\}$$

(assuming that the product of the expected excess return and the expected growth rate of per capita consumption is small).

This way of expressing the equity risk premium shows that it depends on two factors:

1. Relative risk aversion.

2. The covariance of the excess return with the growth rate of per capita consumption.

We may now get a handle on the famous "equity premium puzzle" of Raj Mehra and Ed Prescott (in the *Journal of Monetary Economics*, March 1985). They use 1870-1979 U.S. data, in which the standard deviation of annual per capita consumption growth is 0.036 (surely an overestimate, based on Christina Romer's famous study of prewar U.S. macro data); that of the excess equity return 0.167 (including the Great Depression); the correlation coefficient between equity excess returns and consumption growth is 0.4; and the realized long-run average equity return premium is 0.062 per annum. What degree of risk aversion is needed to rationalize this? Solve for R using

$$0.062 = R \times 0.4 \times 0.036 \times 0.167.$$

Because consumption growth is so smooth for the United States, the answer of $R = 25.8$ is much larger than most economists would regard as reasonable.

Several potential solutions have been suggested. One now in vogue is the possibility of some catastrophic negative shock, whose likelihood is understated by reliance on historical probability frequencies (such as a recent paper by Robert J. Barro, "Rare Disasters and Asset Markets in the Twentieth Century," *Quarterly Journal of Economics,* August 2006).

Now let's look at multiperiod equity pricing. Let us recall the Euler equation,

$$q_t = \mathrm{E}_t\left\{\frac{\beta u'(c_{t+1})}{u'(c_t)}\left(d_{t+1} + q_{t+1}\right)\right\},$$

and substitute recursively to get

$$q_t = \mathrm{E}_t \left\{ \frac{\beta u'(c_{t+1})}{u'(c_t)} d_{t+1} + \frac{\beta^2 u'(c_{t+2})}{u'(c_t)} d_{t+2} + \frac{\beta^2 u'(c_{t+2})}{u'(c_t)} q_{t+2} \right\}.$$

Going to the limit, we find that

$$q_t = \mathrm{E}_t \left\{ \sum_{i=1}^{\infty} \frac{\beta^i u'(c_{t+i})}{u'(c_t)} d_{t+i} \right\} + \lim_{i \to \infty} \mathrm{E}_t \frac{\beta^i u'(c_{t+i})}{u'(c_t)} q_{t+i},$$

and if we assume the transversality condition that $\lim_{i\to\infty} \mathrm{E}_t \beta^i q_{t+i} u'(c_{t+i})/u'(c_t) = 0$,

$$q_t = \sum_{i=1}^{\infty} \mathrm{E}_t \left\{ \frac{\beta^i u'(c_{t+i})}{u'(c_t)} d_{t+i} \right\}. \tag{5}$$

This is the analog of equation (2) for the model with risk aversion. (This PDV relation is true for any individual's consumption.)

There is another interpretation of this condition that makes the comparison with equation (2) clearer. Define $R_{t,t+i}$ to be the price, in terms of date t's output, of a unit of output delivered with certainty on date $t+i$. If the real interest rate is constant at r, then $R_{t,t+i} = 1/(1+r)^i$. In general, $R_{t,t+i}$ is the inverse of the *long-term interest rate* between dates t and $t+i$. The usual logic of Euler equations tells us that in equilibrium,

$$R_{t,t+i} = \mathrm{E}_t \left\{ \frac{\beta^i u'(c_{t+i})}{u'(c_t)} \right\}.$$

Now use the decomposition we invoked earlier to rewrite (5) as

$$q_t = \sum_{i=1}^{\infty} R_{t,t+i} \mathrm{E}_t \{d_{t+i}\} + \sum_{i=1}^{\infty} \mathrm{Cov}_t \left\{ \frac{\beta^i u'(c_{t+i})}{u'(c_t)}, d_{t+i} \right\}.$$

The stock price can be expressed as the PDV (at market interest rates) of expected future dividends – as in the risk-neutral pricing model – plus a risk correction. If consumption tends to be positively conditionally correlated with dividends, the stock price is depressed relative to the PDV model, and in the opposite case, it is raised. Of course, a lower stock price, all else equal, implies a higher expected rate of return.

More on Rational Bubbles

For this section let's again denote the price in (5) by q_t^*. This price obviously satisfies the Euler equation for each date,

$$q_t^* = \mathrm{E}_t \left\{ \frac{\beta u'(c_{t+1})}{u'(c_t)} \left(d_{t+1} + q_{t+1}^* \right) \right\}.$$

Are there other solutions? Let $\tilde{q}_t = q_t^* + b_t$. For this to be a solution, the variables $\{b_t\}$ must satisfy

$$b_t = \mathrm{E}_t \left\{ \frac{\beta u'(c_{t+1})}{u'(c_t)} b_{t+1} \right\}.$$

Mathematically, there can be many types of bubble. The simplest might be to specify

$$b_t = \frac{k}{\beta^t u'(c_t)}$$

for any constant k. Then

$$\begin{aligned} \mathrm{E}_t \left\{ \frac{\beta u'(c_{t+1})}{u'(c_t)} b_{t+1} \right\} &= \mathrm{E}_t \left\{ \frac{\beta u'(c_{t+1})}{u'(c_t)} \frac{k}{\beta^{t+1} u'(c_{t+1})} \right\} \\ &= b_t. \end{aligned}$$

Clearly, because $\beta < 1$, this bubble will tend to explode over time.

To take a more subtle example proposed by Olivier Blanchard in *Economics Letters* (1979), imagine that our bubble has the form

$$b_t = \begin{cases} \frac{k}{\pi^t \beta^t u'(c_t)} & \text{(with probability } \pi) \\ 0 & \text{(with probability } 1 - \pi) \end{cases}$$

conditional on $b_{t-1} > 0$; but if $b_{t-1} = 0, b_t = 0$ with probabilty 1. The transition probabilities are independent of the rest of the economy. Then we once again have a bubble because if $b_t > 0$,

$$\begin{aligned} \mathrm{E}_t \left\{ \frac{\beta u'(c_{t+1})}{u'(c_t)} b_{t+1} \right\} &= \pi \mathrm{E}_t \left\{ \frac{\beta u'(c_{t+1})}{u'(c_t)} \frac{k}{(\pi\beta)^{t+1} u'(c_{t+1})} \right\} + (1 - \pi) \cdot 0 \\ &= \frac{k}{(\pi\beta)^t u'(c_t)} = b_t. \end{aligned}$$

This is a bubble that "crashes" permanently to 0 with probability $1-\pi$, and so it grows faster prior to the crash. An interesting (and realistic) feature of this crashing bubble is that it must crash in finite time with probability 1.

The problem set contains an even weirder example, and discusses arguments for excluding rational bubbles of this kind on theoretical grounds (at least in nonmonetary models). Note that the uniqueness-of-equilibrium proof given in the Lucas (1978) paper does not cover the possibility of rational stock-price bubbles. (Lucas proves uniqueness only within a class of pricing functions that does not admit potentially unbounded bubbles.)

Economics 202A: Macroeconomics

Problem Set 6

Due: Date to be determined

1. *Capital asset pricing model ("Classic" CAPM).* Assume a two-period model in which each investor i maximizes the expected value of a quadratic function of next period's random wealth, W^i:

$$U^i = \mathrm{E}\left\{\alpha W^i - \frac{\gamma}{2}\left(W^i\right)^2\right\}.$$

Next period's wealth W^i depends on current wealth, W_0^i, the realized (net) returns on the N risky assets in which wealth can be held, $\{r_j\}_{j=1}^N$, the nonrandom return r_F on a riskless asset, and the investment shares $\left\{x_j^i\right\}_{j=1}^N$ of initial wealth that investor i selects for the available risky assets:

$$W^i = W_0^i\left[\sum_{j=1}^N x_j^i(1+r_j) + \left(1 - \sum_{j=1}^N x_j^i\right)(1+r_F)\right].$$

(a) Show how to write the last equation as

$$W^i = W_0^i\left[(1+r_F) + \sum_{j=1}^N x_j^i(r_j - r_F)\right].$$

(b) Derive investor i's first-order optimum condition with respect to x_j^i.

(c) Sum these optimum conditions over all M investors i to derive an equilibrium condition involving aggregate second-period wealth, $W \equiv \sum_{i=1}^M W^i$.

(d) Define the coefficient

$$\rho \equiv \frac{\gamma \mathrm{E}\{W\}/M}{\alpha - \gamma \mathrm{E}\{W\}/M}.$$

Show that we can interpret ρ as a measure of "average" relative risk aversion.

(e) Define the (gross) "return on the market" as

$$1 + r_M = \frac{W}{W_0},$$

where $W_0 \equiv \sum_{i=1}^{M} W_0^i$. Show that the equilibrium condition from part c, above, can be put into the form:

$$\mathrm{E}\left\{r_j - r_F\right\} = \frac{\rho \mathrm{Cov}\left\{r_j, r_M\right\}}{\mathrm{E}\left\{1 + r_M\right\}}.$$

(f) What is the intuitive interpretation of the last condition?

(g) Show how to write the condition from part e, above, as

$$\mathrm{E}\left\{r_j - r_F\right\} = \beta_j \mathrm{E}\left\{r_M - r_F\right\},$$

where

$$\beta_j \equiv \frac{\mathrm{Cov}\left\{r_j, r_M\right\}}{\mathrm{Var}(r_M)}.$$

The CAPM framework predicts that a risky asset's "beta," as defined here, determines the degree to which it can be expected to outperform the market as a whole. (This form of the model can be tested from market returns data alone, without assumptions on the degree of risk aversion.)

□

Economics 202A, Problem Set 5

Maurice Obstfeld

1. *Consumer durables.* Our consumption analysis implicitly assumed that consumption is perishable. But if some consumer goods instead were durable (washing machines, autos, etc.), spending in one period would secure an item that yields utility in a number of subsequent periods. This question asks that you analyze consumer behavior when some goods are durable.

Before going further, here is a digression on a solution method, the method of *lag and lead operators*, that should be in your toolkit. For any time series $\{x_t\}$, define the lag operator L by

$$Lx_t = x_{t-1}.$$

Define the lead operator L^{-1} by

$$L^{-1}x_t = x_{t+1}.$$

(Obviously, $LL^{-1}x_t = L^{-1}Lx_t = x_t$, which fact inspires the notation.) You need to know two facts about these operators, both derived from the standard formula for summing geometric series.

FACT #1. Let $y_t = (1 - \theta L)x_t$, where $|\theta| < 1$. Then

$$x_t = (1 - \theta L)^{-1}y_t,$$

where (note the formal similarity to the usual formula)

$$(1 - \theta L)^{-1} = 1 + \theta L + \theta^2 L^2 + \theta^3 L^3 + \; ....$$

Observe that $(1 - \theta L)\left(1 + \theta L + \theta^2 L^2 + \theta^3 L^3 + \; ...\right) = 1$.

FACT #2. Let $y_t = (1 - \theta L)x_t$, where $|\theta| > 1$. Then

$$x_t = -\theta^{-1}L^{-1}(1 - \theta^{-1}L^{-1})^{-1}y_t,$$

where

$$(1 - \theta^{-1}L^{-1})^{-1} = 1 + \theta^{-1}L^{-1} + \theta^{-2}L^{-2} + \theta^{-3}L^{-3} + \; ....$$

To establish this last formula, note that

$$\left(1-\theta^{-1}L^{-1}\right)\left(1+\theta^{-1}L^{-1}+\theta^{-2}L^{-2}+\theta^{-3}L^{-3}+\ \ldots.\right)=1.$$

It is *not* correct to write $x_t = (1-\theta L)^{-1}y_t$ in this case because $|\theta| > 1$, which means that $(1-\theta L)^{-1}$ does not exist. However, if $|\theta| > 1$, then $\left|\theta^{-1}\right| < 1$, and $y_t = (1-\theta L)x_t = \theta L\left(\theta^{-1}L^{-1}-1\right) = -\theta L\left(1-\theta^{-1}L^{-1}\right)x_t$. Because $1-\theta^{-1}L^{-1}$ *is* invertible when $\left|\theta^{-1}\right| < 1$, we can therefore write $x_t = -\theta^{-1}L^{-1}(1-\theta^{-1}L^{-1})^{-1}y_t$.

Now, the model of durables. An individual maximizes

$$\sum_{t=0}^{\infty}\beta^t\left[u(c_t)+v(s_t)\right]$$

where c_t is nondurable consumption and s_t is the stock (measured at the start of period t) of a consumer durable yielding a flow of services proportional to s_t. Let z_t be purchases of durables (which may be resold on a secondary market): if durables do not depreciate, then

$$s_{t+1} = s_t + z_t.$$

Let a_t be the value (in terms of consumption c, at the start of period t) of the individual's financial assets, which have a constant real gross per period yield of $1+r$. If (for simplicity) we assume that the durable good's price in terms of c is constant at 1, then (make sure you see why)

$$a_{t+1} = (1+r)a_t + y_t - c_t - z_t,$$

where y_t is an exogenous flow of income.

(a) Using any method you wish, derive and *interpret* the following first-order conditions for the consumer's problem:

$$u'(c_t) = \beta(1+r)u'(c_{t+1}),$$

$$u'(c_t) = \beta u'(c_{t+1}) + \beta v'(s_{t+1}).$$

(b) Write the second f.o.c. above as $-v'(s_t) = (1-\beta^{-1}L)u'(c_t)$ and show that $u'(c_t) = \beta v'(s_{t+1}) + \beta^2 v'(s_{t+2}) + \ \ldots$, using the preceding lag-lead formalism. Interpret this condition.

(c) Show that the equation

$$v'(s_t) = \beta(1 + r)v'(s_{t+1})$$

holds at the individual optimum. Thus, when $\beta(1 + r) = 1$, the consumer will smooth the marginal utility of durable services.

(d) What does this finding imply about the smoothness of *total* spending $c + z$? To think about that question, let $\beta(1 + r) = 1$ and assume that $u(c) = \gamma \log(c)$, $v(s) = (1 - \gamma) \log(s)$. Then solve explicitly for the paths of c, z, and s.

(e) How would the problem change if we allowed explicitly for a rental market in durables?

2. *The Lucas "tree" model from* Econometrica*, 1978.* Consider a world with a single representative agent, in which a random and exogenous amount of perishable output y_t falls from a fruit tree each period t. (There is no other output.) Output follows the stochastic process

$$\log y_t = \log y_{t-1} + \varepsilon_t, \qquad \mathrm{E}_{t-1}\varepsilon_t = 0, \tag{1}$$

where the i.i.d. shock ε_t is drawn from a $N(0, \sigma^2)$ normal distribution. There is no way to grow more fruit trees — the supply is fixed.

The agent's lifetime utility function is

$$\mathbb{E}_t \left\{ \sum_{s=t}^{\infty} \mathrm{e}^{-\theta(s-t)} u(c_s) \right\},$$

where $\theta > 0$ is the rate of time preference. Assume that there is a competitive stock market in which people can trade shares in the fruit tree, whose price on date t is p_t. This is the ex dividend price: if you buy a share on date t, you get your first dividend on date $t + 1$.

(a) Show that an individual will choose contingent consumption plans such that on each date.

$$p_t u'(c_t) = \mathrm{e}^{-\theta} \mathbb{E}_t \left\{ (y_{t+1} + p_{t+1}) u'(c_{t+1}) \right\}. \tag{2}$$

(You can use the individual finance constraint that $c_t + p_t x_{t+1} \leq (y_t + p_t)\, x_t$, where x_t is the share of the fruit tree the individual holds at the end of period $t - 1$.)

(b) Show that in equilibrium, the "fundamentals" price of the tree is

$$p_t^* = \mathbb{E}_t \left\{ \sum_{s=t+1}^{\infty} \mathrm{e}^{-\theta(s-t)} \frac{u'(y_s)\, y_s}{u'(y_t)} \right\}.$$

Can you interpret this price in terms of expected dividends and risk factors? What is the sign of these risk factors on the trees?

(c) Let $u(c) = c^{1-\gamma}/(1-\gamma)$ for $\gamma > 0$. Show that the normality assumption in (1) implies (for $s > t$):

$$\mathbb{E}_t \left\{ y_s^{1-\gamma} \right\} = y_t^{1-\gamma} \mathrm{e}^{\frac{\sigma^2(1-\gamma)^2}{2}(s-t)}.$$

(Use the *lognormal* distribution: if $\varepsilon \sim N(\mu, \sigma^2)$, then e^{ε} has a lognormal distribution with $\mathbb{E}\left\{\mathrm{e}^{\varepsilon}\right\} = \mathrm{e}^{\mu + \frac{1}{2}\sigma^2}$.)

(d) Deduce from part (c) that if $\theta > \sigma^2(1-\gamma)^2/2$, then $p_t^* = \kappa y_t$, where

$$\kappa = \frac{1}{\left\{\mathrm{e}^{[\theta - \sigma^2(1-\gamma)^2/2]} - 1\right\}}.$$

(e) Now return to a general strictly concave utility function $u(c)$. Let b_t be the random variable $A y_t^{\rho}/u'(y_t)$, where $\rho = \sqrt{2\theta/\sigma^2}$ and A is an arbitrary constant. Show that $p_t^* + b_t$ will satisfy the individual's Euler equation (2) in equilibrium, so that b_t is a bubble.

(f) Show that $p_t = p_t^* + b_t$ violates the (equilibrium) transversality condition:

$$\lim_{T \to \infty} \mathrm{e}^{-\theta(T-t)} \mathbb{E}_t \left\{ u'(y_{t+T}) p_{t+T} \right\} = 0. \tag{3}$$

(g) Together with the equilibrium Euler equation [equation (2) with y substituted for c],

$$p_t u'(y_t) = \mathrm{e}^{-\theta} \mathbb{E}_t \left\{ (y_{t+1} + p_{t+1}) u'(y_{t+1}) \right\}, \tag{4}$$

condition (3) is *sufficient* for a stochastic price path $\{p_t\}$ to be an equilibrium of the Lucas model. In this part of the homework we will show that the condition is also *necessary.*

Iterate (4) forward to derive

$$p_t u'(y_t) = \mathbb{E}_t \left\{ \sum_{s=t+1}^{\infty} \mathrm{e}^{-\theta(s-t)} u'(y_s)\, y_s \right\} + \lim_{T \to \infty} \mathrm{e}^{-\theta(T-t)} \mathbb{E}_t \left\{ u'(y_{t+T}) p_{t+T} \right\}.$$

Argue that free disposal of output ensures that the limit in condition (3) must be nonnegative. Argue that if the limit is strictly positive, we cannot be looking at an equilibrium because individuals can raise expected lifetime utility through the following strategy: sell a tiny amount of the fruit tree today and consume the proceeds now, never repurchasing the portion of the fruit tree just sold (that is, reduce x_t, which equals 1 in the hypothesized equilibrium, permanently). Why is the Euler equation (4) alone not sufficient to rule out the possibility that such a strategy raises lifetime utility?